



The cyclicity of a hypergraph

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Abstract

The cyclicity of a hypergraph is an efficiently computable integer that extends the notion of the cyclomatic number of a graph. Generalizing the notion of the degree of a node in a graph, we define the star articulation degree of a subedge in a hypergraph, and then use it to set up the expression for the cyclicity. The basic properties of cyclicity are that it is zero on acyclic hypergraphs and strictly positive otherwise, and that on graphs it coincides with the cyclomatic number; moreover, the cyclicity depends only on maximal edges, decreases on subhypergraph, and is additive on compositions. We introduce the notions of circulant graphs and join-graphs of a hypergraph. Neither of these two kinds of graphs is uniquely determined by a hypergraph; however, every circulant graph and every join-graph of a hypergraph has the cyclomatic number equal to the cyclicity of the hypergraph. We also compare the cyclicity of a hypergraph with the cyclomatic number of a hypergraph, which is another, already known, extension of the cyclomatic number of a graph.

1. Introduction

Extensions of the cyclomatic number of a graph to hypergraphs are nothing new. Berge gives in [3] one such extension, the cyclomatic number $\mu(E)$ of a hypergraph E , introduced by Acharya and Las Vergnas in [1]. It is defined as

$$\mu(E) := \sum_{e \in E} |e| - |\bigcup E| - w_E,$$

where w_E is the maximal weight of a subforest in the intersection graph of the hypergraph E . The weight is calculated as the sum of terms $|e_1 \cap e_2|$ over all arcs $\{e_1, e_2\}$ of the subforest. The cyclomatic number $\mu(E)$ is non-negative and is zero precisely when the hypergraph E is acyclic; moreover, when E is a graph, $\mu(E)$ is the usual cyclomatic number of the graph E .

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In this paper we propose another integral measure, the cyclicity γ , defined for hypergraphs, which has the same key properties as the cyclomatic number μ : it extends the notion of the cyclomatic number for graphs, is always non-negative, and is zero precisely on acyclic hypergraphs. The cyclicity of a hypergraph E is defined as

$$\gamma(E) := \sum_f (\delta_E(f) - 1) - |\text{Max}(E)| + 1,$$

where $\delta_E(f)$ is the ‘star articulation degree’ of a subedge f in E and the sum runs over all those subedges f that have this degree at least two.

What use is another extension of the cyclomatic number of graphs to hypergraphs? One interesting point of $\gamma(E)$ is its form, as a simple additive combination of terms reflecting local structural properties of the hypergraph E . By contrast, the term w_E in the expression for the cyclomatic number $\mu(E)$ is decidedly a global characteristics of the hypergraph E . The cyclicity has, in addition to this rather technical distinction, some properties not enjoyed by the cyclomatic number, e.g. the cyclicity is preserved under blowups of a hypergraph while the cyclomatic number is not.

The cyclicity of hypergraphs is related to the cyclomatic number of graphs in another way. We can associate with a hypergraph certain ‘circulant graphs’ so that the cyclomatic number of each of these graphs is equal to the cyclicity of the hypergraph. Another kind of graphs associated with a hypergraph are its join-graphs, which are natural generalizations of join-trees of an acyclic hypergraph. We will be able to completely analyze the structure of join-graphs, thus gaining an insight into the meaning of summands in the formula for cyclicity. This analysis will show, in particular, that every join-graph of a hypergraph has the cyclomatic number equal to the cyclicity of the hypergraph.

In spite of differences between the cyclomatic number and the cyclicity of a hypergraph they parallel each other in several respects. Both are efficiently computable, both depend only on the maximal edges, both decrease on passing to subhypergraphs, and both are additive on compositions.

Here is a short overview of the contents. Section 2 gives basic definitions. In Section 3 we define the star articulation degree of a subedge in a hypergraph and consider the joints of a hypergraph, which are the subedges whose degree is at least two. After these preparations we define in Section 4 the cyclicity of a hypergraph and prove its key properties. In Sections 5 and 6 we introduce and examine circulant graphs and join-graphs associated with a hypergraph. We compare the cyclicity of a hypergraph with the cyclomatic number of a hypergraph in Section 7, and conclude with some reflections on the cyclicity and related notions.

2. Preliminaries

A *hypergraph* is any finite collection of *edges*, where each edge is a finite set of *vertices*. We allow a hypergraph to have an empty edge. The *span* of a hypergraph E is

the union $\cup E$ of its edges; its elements are called the *vertices* of E . A *subedge* of a hypergraph E is a subset of an edge of E . We denote by $\text{Max}(E)$ the hypergraph consisting of all maximal edges of a hypergraph E . A hypergraph whose edges are pairwise incomparable is said to be *simple*. A subset of a hypergraph E is called a *partial hypergraph* of E .

Let E be a hypergraph and U any set of vertices of E . We call the hypergraph

$$E(U) := \{e \subseteq U \mid e \in E\}$$

the *part of E in U* , and the hypergraph

$$E[U] := \{e \cap U \mid e \in E\}$$

the *subhypergraph induced by E in U* . If $U \supseteq V$ are any two sets of vertices of E , then $(E(U))(V) = E(V)$ and $(E[U])[V] = E[V]$. The operation of inducing the subhypergraph in a fixed vertex set U has the property that $\text{Max}(E[U])$ depends only on $\text{Max}(E)$; the operation of taking the part of a hypergraph in a fixed vertex set does not have this property.

With E and U as in the previous paragraph, we construct the *intersection graph of E relative to U* , which is the graph G with the set of nodes $E \setminus E(U)$, whose arcs are all pairs $\{e, f\}$ of different edges of E such that $e \cap f$ is not a subset of U . Node sets of the connected components of the intersection graph G are the *connected components of E relative to U* ; the connected components of E relative to the empty set are simply the *connected components of E* .

We shall also be dealing with graphs, always undirected and without loops or multiple edges. To distinguish graphs from hypergraphs, we will talk of a graph as of a set of *nodes* connected by a set of *arcs*. A graph G can be regarded as a simple hypergraph, consisting of the two-element edges $\{u, v\}$ for the arcs uv of G , and of the singletons $\{u\}$ for the isolated nodes u of G .

Finally, let us mention acyclic hypergraphs, just for reference. There are many characterizations of acyclic hypergraphs (Beeri et al. give in [2] quite a few). We will present here only one of them, as the definition of acyclic hypergraphs. A *join-tree* is a tree T which has for the set of nodes a simple hypergraph E and satisfies the following condition: for any three nodes $e_0, e, e_1 \in E$, with the node e lying on the unique path in the tree T between the nodes e_0 and e_1 , we have $e \supseteq e_0 \cap e_1$. Now a simple hypergraph E is said to be *acyclic* if it admits a join-tree, i.e. if there exists a join-tree on the set of nodes E .

3. Star articulation degree and joints

Let E be a hypergraph. For any set of vertices U , the *star of U in E* , notation $\text{St}_E(U)$, is the hypergraph consisting of all edges $e \in E$ such that $U \subseteq e$. (the star is nonempty if and only if U is a subedge of E). Let now f be a subedge of the hypergraph E . The *star articulation degree of f in E* , denoted $\delta_E(f)$, is the number of connected components of

the star $\text{St}_E(f)$ relative to the subedge f ; since $\delta_E(f)$ is the only kind of degree considered in this paper, we will refer to it simply as the *degree of f in E* . The subedge f has the same degree in E as in $\text{Max}(E)$. Maximal edges of E have degree zero, while a proper subedge of E has degree at least one. Call the subedge f whose degree in E is at least two, a *joint of E* , and let $\Psi(E)$ denote the set of all joints of E . Joints of a hypergraph generalize those nodes of a graph which have at least two neighbours.

Lemma 1. *Every joint of a hypergraph E is the intersection of two different maximal edges of E .*

Proof. Let f be a joint of E . Take any two maximal edges e_1 and e_2 of E from different connected components of the star $\text{St}_E(f)$ relative to f . Then clearly $e_1 \cap e_2 = f$. \square

A simple but important consequence of this lemma is the upper bound $|E|(|E| - 1)/2$ on the number of joints of a hypergraph E .

For any hypergraph E let $\Lambda(E)$ denote the set of all intersections of pairs of different maximal edges of E . We have just shown that $\Psi(E)$ is a subset of $\Lambda(E)$. We can say more:

Lemma 2. $\text{Max}(\Lambda(E)) = \text{Max}(\Psi(E))$.

Proof. Let f be maximal in $\Lambda(E)$. Any two different maximal edges of E that include f must have intersection f , by maximality of f . This means that each maximal edge of the star $\text{St}_E(f)$ belongs to a different connected component of the star relative to f , and since there are at least two different maximal edges in the star, the subedge f is a joint of E , clearly a maximal one.

Conversely, let f be a maximal joint. Since f belongs to $\Lambda(E)$, it is a subset of some maximal member g of $\Lambda(E)$. Now according to the first part of the proof g is a joint, hence must be equal to the maximal joint f . \square

An easy consequence is that any hypergraph with at least two different maximal edges has joints. As we can see from the proof, a joint f of a simple hypergraph E is maximal precisely when each connected component of the star $\text{St}_E(f)$ relative to f consists of a single edge.

4. The cyclicity of a hypergraph

In this section we introduce the cyclicity of a hypergraph and prove two of its key properties, namely that it decreases on subhypergraphs and that it extends the cyclomatic number of graphs.

Definition 3. Let E be any hypergraph. The integer

$$\gamma(E) := \sum_{f \in \Psi(E)} (\delta_E(f) - 1) - |\text{Max}(E)| + 1$$

will be called the cyclicity of E .

Sometimes it comes handy to have the summation in the definition of $\gamma(E)$ extended over some other proper subedges of E besides the joints; this does not change the sum, since all additional terms are zeros. We can also incorporate the term $|\text{Max}(E)|$ into the sum and write

$$\gamma(E) := \sum_f (\delta_E(f) - 1) + 1$$

with f running over all maximal edges and all joints, and perhaps over some other subedges. From the defining formula it is clear that the cyclicity is efficiently computable (assuming the hypergraph E is given by an explicit list of edges). Because each subedge has the same degree in E as in $\text{Max}(E)$, we have $\gamma(E) = \gamma(\text{Max}(E))$.

Theorem 4. If F is a subhypergraph of E , then $\gamma(F) \leq \gamma(E)$.

Proof. We can assume that E is simple and that $F = E[U]$, where U is the span $\bigcup E$ minus some vertex u of E . The hypergraph E can be written as the disjoint union $E = E(U) \cup \text{St}_E(u)$. Put $S := \text{St}_E(u)$ and

$$S_0 := \{e \in S \mid e \setminus \{u\} \text{ is a subedge of } E(U)\}$$

and let S_1 be the complement of S_0 in S . Note that for every $e_0 \in S_0$, $e_0 \setminus \{u\}$ is a proper subedge of $E(U)$. Then

$$\text{Max}(F) = E(U) \cup S_1[U],$$

where the union is disjoint and different edges e from S_1 give rise to different edges $e \cap U = e \setminus \{u\}$ of $S_1[U]$, so that

$$|\text{Max}(F)| = |E| - |S_0|.$$

We must show that in passing from E to F , the sum in the formula for the cyclicity (with summation over all proper subedges) decreases by at least $|S_0|$.

Let f be any proper subedge of F . If f is not a subset of any edge of S , then $\text{St}_E(f) = \text{St}_F(f)$, hence $\delta_F(f) = \delta_E(f)$. Now assume that f is a subset of at least one edge of S , and write $R := \text{St}_S(f)$, $S' := \text{St}_{E(U)}(f)$. Then $\text{St}_E(f) = S' \cup R$ and $\text{St}_F(f) = S' \cup R[U]$. All the edges of R are directly connected to each other relative to f , because the intersection of any two of them contains the vertex $u \notin f$. Suppose that $f \notin R[U]$. Then the connected components of $\text{St}_F(f)$ relative to f which contain edges $e' \in R[U]$, say there are k such components, get merged into a single connected

component of $\text{St}_E(f)$ relative to f , while all other connected components relative to f are the same in $\text{St}_F(f)$ as in $\text{St}_E(f)$. Thus $\delta_E(f) = \delta_F(f) - k + 1$. But now we have also a proper subedge $f \cup \{u\}$ of E , where $\delta_E(f \cup \{u\}) \geq k$, whence $\delta_E(f) + \delta_E(f \cup \{u\}) \geq \delta_F(f) + 1$, i.e.,

$$(\delta_E(f) - 1) + (\delta_E(f \cup \{u\}) - 1) \geq \delta_F(f) - 1.$$

There remains the case with $f \in R[U]$. Since f is a proper subedge of F , we must have $f = e_0 \setminus \{u\}$ for some $e_0 \in S_0$. Moreover, $R = \{e_0\}$ and the connected components of $\text{St}_E(f)$ relative to f are those of $\text{St}_F(f)$ and the one additional component $\{e_0\}$, therefore $\delta_E(f) = \delta_F(f) + 1$. We have already noticed that, for every $e_0 \in S_0$, $e_0 \setminus \{u\}$ is a proper subedge of $E(U)$ and hence of F . This gives us $|S_0|$ proper subedges of F at each of which the sum in the definition of cyclicity gains 1 on going from F to E . \square

Since the hypergraph whose only edge is the empty set is a subhypergraph of every hypergraph, we have the following.

Corollary 5. *The cyclicity of a hypergraph is non-negative.*

We shall see later on (Theorem 12) that the cyclicity of a simple hypergraph is zero precisely when the hypergraph is acyclic.

Lemma 6. *Let G be an undirected graph represented as a simple hypergraph. Then $\gamma(G)$ is the usual cyclomatic number of the graph G .*

Proof. Let G have q arcs, p nodes, and s connected components. Some nodes may be isolated, say p_0 of them. We extend the sum in the definition of cyclicity over all proper subedges of G , which are the singletons $\{u\}$ for all non-isolated nodes u and the empty subedge. Computing the cyclicity of G ,

$$\begin{aligned} \gamma(G) &= \sum_u (\delta(u) - 1) + (\delta(\emptyset) - 1) - |G| + 1 \\ &= 2q - (p - p_0) + (s - 1) - (q + p_0) + 1 = q - p + s, \end{aligned}$$

we find that it is in fact equal to the cyclomatic number of the graph G . \square

5. Circulant graphs

We will associate with a hypergraph certain ‘circulant’ graphs. There may be several circulant graphs associated with the hypergraph, but they all have the same cyclomatic number, which is equal to the cyclicity of the hypergraph.

Let E be a simple hypergraph. Add to E all its joints and perhaps some other subedges of E , and denote the resulting hypergraph by F . (Hypergraphs assembled

in this manner are characterized by the property that all of their joints are their edges.) We take the hypergraph F for the node set of a graph G . For each edge f of F which is a proper subedge of E , and each connected component C of the star $\text{St}_F(f)$ relative to f , we choose an edge f' in C ; the arcs of G are then all the pairs $\{f, f'\}$. We will say that any graph G obtained in this way is a *circulant graph* of a simple hypergraph E .

Theorem 7. *Every circulant graph G of a simple hypergraph E is connected, and the cyclomatic number of G is equal to the cyclicity of E .*

Proof. Take a look at the formula for the cyclicity of E (Definition 3). We can assume that the sum of terms $\delta_E(f) - 1$ runs over all those edges f in the node set F of the graph G that are proper subedges of E . Since $E = \text{Max}(F)$, we have $\delta_E(f) = \delta_F(f)$ for every edge f of F . Now split the sum into the difference of the sum of degrees $\delta_F(f)$, which is precisely the number of arcs of the graph G , and the sum of 1's, which together with the term $|E|$ yields the number of nodes of G . We see that the whole expression in fact gives the cyclomatic number of the graph G , provided G is connected.

Suppose G is not connected. Then the node set F can be partitioned into two non-empty subsets F_1 and F_2 such that each arc of G lies within one of these two subsets. Since every node of G is connected in G to some node in E , the sets $E_1 := E \cap F_1$ and $E_2 := E \cap F_2$ are not empty. Let g be maximal among all intersections of the form $e_1 \cap e_2$ with $e_1 \in E_1$ and $e_2 \in E_2$; because e_1 and e_2 are different maximal edges of F , g is a proper subedge of both e_1 and e_2 . The star $S := \text{St}_F(g)$ is partitioned into $S_1 := S \cap F_1$ and $S_2 := S \cap F_2$, where $e_1 \in S_1$ and $e_2 \in S_2$. The intersection of any edge from S_1 with any edge from S_2 is g , because of the maximality of g . This means that the star S has at least two connected components relative to g , so g is a joint and therefore belongs to F , say $g \in F_1$. But then g is connected by an arc of G to some node in S_2 , a contradiction. \square

In particular, the cyclicity of a simple hypergraph is zero if and only if some of its circulant graphs is a tree — in which case every circulant graph of the hypergraph is a tree.

6. Join-graphs, the join-invariant, and the cyclicity

Let E be a simple hypergraph and G a graph on the set of nodes E . If f is a subedge of E , then a walk in G whose every node includes f will be said to be *over f* . We will say that G *joins a pair of edges e_1, e_2 of E* if and only if the edges e_1 and e_2 are connected in G by a walk over $e_1 \cap e_2$, and will say that G *joins the hypergraph E* if and only if G joins every pair of edges of E . A graph G that is minimal (as a set of arcs) w.r.t. the property of joining the hypergraph E , will be called a *join-graph* of E .

The complete graph on the set of nodes E clearly joins E ; it follows that the hypergraph E has at least one join-graph. Suppose that a graph G joins E , and that $a = \{e_1, e_2\}$ is an arc of G such that the edges e_1 and e_2 are joined in G by a walk p on which the arc a does not appear; let us call such an arc a of the joining graph G *redundant*. If we remove from G a redundant arc $a = \{e_1, e_2\}$, then the remaining graph G' still joins the hypergraph E : any pair of edges d_1, d_2 of E is joined by a walk in G ; if the walk contains the arc a , then $e_1 \cap e_2 \supseteq d_1 \cap d_2$, so substituting the walk p for the arc a we obtain a walk in G' over $d_1 \cap d_2$, which therefore joins d_1 and d_2 . On the other hand, if no arc of the graph G is redundant, and we remove from it one or more arcs, then the end nodes of any of the removed arcs can no longer be joined in the remaining graph. Thus we have

Lemma 8. *A graph that joins a simple hypergraph is its join-graph if and only if it does not have any redundant arcs.*

We will now state the structure theorem for join-graphs. We need some notions to do this. A pair $\{e_1, e_2\}$ of different edges of a simple hypergraph E is said to be *articulated* if and only if e_1 and e_2 are not connected in the star $\text{St}_E(e_1 \cap e_2)$ relative to $e_1 \cap e_2$. Equivalently, $\{e_1, e_2\}$ is an articulated pair if and only if $e_1 \cap e_2 := f$ is a joint and the edges e_1 and e_2 lie in different connected components of the star $\text{St}_E(f)$ relative to f . For any joint f of E we denote by $\mathcal{C}_E(f)$ the set of all connected components of $\text{St}_E(f)$ relative to f . Let f be a joint and T a set of articulated pairs $\{e_1, e_2\}$ with $e_1 \cap e_2 = f$. To each pair $\{e_1, e_2\}$ in T there corresponds the pair $\{C_1, C_2\}$ of different connected components in $\mathcal{C}_E(f)$, where C_i is the connected component containing the edge e_i , for $i = 1, 2$; let \mathcal{T} be the set of all pairs of connected components corresponding to the pairs of edges in T . We call T a *tree set over f* if and only if the mapping $\{e_1, e_2\} \mapsto \{C_1, C_2\}$ from T to \mathcal{T} is bijective and \mathcal{T} is the set of arcs of a tree on the set of nodes $\mathcal{C}_E(f)$.

Theorem 9 (Structure of join-graphs). *Let G be a join-graph of a simple hypergraph E . All arcs of G are articulated edge pairs in E , and for each joint f of E the set T_f of all arcs $\{e_1, e_2\}$ of G with $e_1 \cap e_2 = f$ is a tree set over f .*

Proof. We show first that any arc $\{e_1, e_2\}$ of G is an articulated pair of edges. Write $f := e_1 \cap e_2$, and assume that e_1 and e_2 are connected in the star $\text{St}_E(f)$ relative to f . Then there is a sequence of edges $h_0 = e_1, h_1, \dots, h_n = e_2$ such that $h_{j-1} \cap h_j \supset f$ for each $j = 1, \dots, n$. Since G joins E , each pair of edges h_{j-1}, h_j is joined in G by a walk p_j . The composition of walks p_1, \dots, p_n is a walk p joining e_1 with e_2 , where the intersection of any two consecutive nodes on p is a proper superset of f . The arc $\{e_1, e_2\}$ does not appear on the walk p and is therefore redundant, contrary to minimality of G . This contradiction shows that e_1 and e_2 are not connected in $\text{St}_E(f)$ relative to f .

Let now f be any joint of E and let the set T_f be as in the statement of the theorem. We will show that there cannot be two different arcs $\{e_1, e_2\}$ and $\{e'_1, e'_2\}$ in T_f such that e_i and e'_i would belong to the same connected component C_i in $\mathcal{C}_E(f)$, for $i = 1, 2$. Assume the contrary. Then the edges e_i and e'_i are joined by a walk p_i in G , $i = 1, 2$ (at least one of the walks p_1, p_2 is of non-zero length), where the intersection of any two consecutive nodes on the walk p_i is a proper superset of the joint f . No arc on p_i can be either $\{e_1, e_2\}$ or $\{e'_1, e'_2\}$, and it follows that, say, the arc $\{e_1, e_2\}$ is redundant. Denote by \mathcal{T}_f the set of all pairs of connected components in $\mathcal{C}_E(f)$ corresponding to arcs in T_f . It can be shown, by an argument similar to the one we have used just now, that \mathcal{T}_f can contain no cycle. It remains to show that \mathcal{T}_f connects any two different nodes C_1 and C_2 in $\mathcal{C}_E(f)$. Choose edges $c_1 \in C_1$ and $c_2 \in C_2$, and let $d_0 = c_1, d_1, \dots, d_n = c_2$ be the sequence of nodes on some walk in G joining c_1 and c_2 ; all the edges d_0, d_1, \dots, d_n belong to $\text{St}_E(f)$. For each $j = 0, 1, \dots, n$ let D_j be the connected component in $\mathcal{C}_E(f)$ containing d_j . Then in the sequence D_0, D_1, \dots, D_n either $D_{j-1} = D_j$, or $\{D_{j-1}, D_j\}$ belongs to \mathcal{T}_f , for each $j = 1, \dots, n$. We have a walk (possibly with some consecutively repeated nodes) connecting C_1 with C_2 in the tree \mathcal{T}_f . \square

The structure theorem for join-graphs has two immediate corollaries. The first corollary asserts the existence of the join-invariant of a simple hypergraph, while the second corollary ‘explains’ the summands in the formula for the cyclicity of a hypergraph as the sizes of tree sets over the joints of a hypergraph.

Corollary 10 (The join-invariant). *Let G be any join-graph of a simple hypergraph E . The multiset (formal linear combination with natural coefficients) of joining sets $e_1 \cap e_2$ for all arcs $\{e_1, e_2\}$ of G is*

$$\sum_f (\delta_E(f) - 1) \cdot f,$$

where the sum is taken over all joints of E .

Corollary 11. *Any join-graph of a simple hypergraph has the cyclomatic number equal to the cyclicity of the hypergraph.*

Join-trees are clearly just join-graphs which happen to be trees. Either all join-graphs of a simple hypergraph are trees, or none is; in the former case, the hypergraph is acyclic and its cyclicity is zero, in the latter it is not acyclic and has a non-zero cyclicity. So we have:

Theorem 12. *A simple hypergraph is acyclic if and only if its cyclicity is 0.*

The structure theorem for join-graphs has an exact converse, the construction theorem, which says in effect that the tree sets determined by a join-graph are arbitrary and independent of each other. We need a preliminary lemma.

Lemma 13. *For any two edges e_1 and e_2 of a simple hypergraph E there exists a sequence of edges starting at e_1 and ending at e_2 , in which each edge is a superset of $e_1 \cap e_2$ and any two consecutive edges form an articulated pair.*

For the proof just join the edges e_1 and e_2 by a walk over $e_1 \cap e_2$ in any join-graph of E . Now we are all set for the construction theorem:

Theorem 14 (Construction of join-graphs). *If E is a simple hypergraph, and for each joint f of E , T_f is any tree set over f , then the union of the tree sets T_f for all joints f is the set of arcs of a join-graph of E .*

Proof. For each tree set T_f denote \mathcal{T}_f the corresponding tree on the node set $\mathcal{C}_E(f)$. The tree sets T_f are disjoint from each other: since for every edge pair $\{e_1, e_2\}$ in T_f we have $e_1 \cap e_2 = f$, tree sets over different joints cannot have any edge pair in common. Let G be the graph on the set of nodes E whose set of arcs is the union of all tree sets T_f . For every joint f of E the set of all arcs $\{e_1, e_2\}$ of G with $e_1 \cap e_2 = f$ is precisely the tree set T_f . If we show that G joins E , it will immediately follow that G is in fact a join-graph of E , since by the structure theorem for join-graphs we cannot remove any arcs from G and still have a graph joining E .

Because of Lemma 13 it suffices to show that an articulated pair of edges $\{e_1, e_2\}$ is joined in G . The proof will be by induction on the joint $f := e_1 \cap e_2$ in the ordering of joints opposite to the inclusion (i.e. the induction will go downwards). Let C_i be the connected component in $\mathcal{C}_E(f)$ containing the edge e_i , $i = 1, 2$. There is a walk $D_0 D_1 \cdots D_n$ ($n \geq 1$) in the tree \mathcal{T}_f connecting $C_1 = D_0$ to $C_2 = D_n$. For each $j = 1, \dots, n$ let $\{d'_{j-1}, d_j\}$ be the pair in the tree set T_j such that $d'_{j-1} \in D_{j-1}$ and $d_j \in D_j$. Here the pairs $\{d'_{j-1}, d_j\}$ are already arcs of the graph G , and the nodes d'_{j-1}, d_j include f . It remains to show that the pairs $\{e_1, d'_0\}, \{d_1, d'_1\}, \dots, \{d_{n-1}, d'_{n-1}\}, \{d_n, e_2\}$ are joined in G . Each of these pairs is connected in the star $\text{St}_E(f)$ relative to f , so let $\{d, d'\}$ be any such pair. There exists a walk $c_0 c_1 \cdots c_m$ connecting $d = c_0$ to $d' = c_m$ (where possibly $m = 0$) in $\text{St}_E(f)$ relative to f . For each $k = 1, \dots, m$ the edges c_{k-1} and c_k are connected, according to Lemma 13, by a walk over $c_{k-1} \cap c_k \supset f$, where all arcs on the walk are articulated pairs of edges. The concatenation of these walks is a walk from d to d' , each arc of which is an articulated pair of edges $\{a, b\}$ with $a \cap b \supset f$, hence by the induction hypothesis the edges a and b are joined in the graph G , and so are then the edges d and d' . \square

A *coherent* join-graph is one in which each tree set T_f is connected, so is in fact a tree naturally isomorphic to the tree \mathcal{T}_f . According to the construction theorem for join-graphs, every simple hypergraph has coherent join-graphs; in particular, every acyclic simple hypergraph has coherent join-trees.

Let us mention, as a curiosity, that the collection \mathcal{M} of all join-graphs of a simple hypergraph E is the set of bases of a binary matroid. We can easily exhibit a coordinatization. Take the set of all pairs (f, C) , where f is a joint of E and C is a

connected component of the star $\text{St}_E(f)$ relative to f , and make this set a basis of a vector space V over the two-element field. Let $a = \{e_1, e_2\}$ be any arc of the complete graph $K(E)$ on the node set E . If $\{e_1, e_2\}$ is an articulated pair, then put $v_a := (f, C_1) + (f, C_2) \in V$, where f is the joint $e_1 \cap e_2$, while C_1 and C_2 are the connected components in $\mathcal{C}_E(f)$ that contain e_1 and e_2 , respectively; otherwise, if the pair $\{e_1, e_2\}$ is not articulated, put $v_a := 0 \in V$. Then a subset G of $K(E)$ (both considered as sets of arcs) is a join-graph of E if and only if the family $(v_b | b \in G)$ is a basis of the subspace of V spanned by the vectors v_a for all arcs a of $K(E)$.

7. Cyclicity vs. cyclomatic number

We have mentioned in the introduction the cyclomatic number $\mu(E)$ of a hypergraph E . In this section we compare it with the cyclicity of a hypergraph.

Let us first verify that the cyclicity is not the same thing as the cyclomatic number. If X is a set of $n \geq 3$ vertices, let E be the hypergraph consisting of edges $X \setminus \{x\}$ for all $x \in X$; then $\gamma(E) = \frac{1}{2}n(n-3) + 1$ and $\mu(E) = n - 2$. For another example take a hypergraph $E = \{a, b, c\}$, where $|(b \cap c) \setminus a| = m$, $|(a \cap c) \setminus b| = n$, and $|(a \cap b) \setminus c| = p$, with $m, n, p > 0$; in this case $\gamma(E) = 1$, while $\mu(E) = \min(m, n, p)$. The two examples demonstrate that the difference $\gamma(E) - \mu(E)$ is not bounded in either direction.

The cyclicity and the cyclomatic number do not differ only in value, they also behave differently. For example, the cyclicity is preserved under blowups, while the cyclomatic number is not. To describe what we mean by a blowup of a hypergraph, let E be a hypergraph and α a surjective function mapping a finite set W onto the span of E . The inverse image $\beta(E)$ of the hypergraph E , consisting of the inverse images $\beta(e) = \alpha^{-1}(e)$ of the edges $e \in E$, is then a *blowup* of E . It is obvious that the maximal edges of the hypergraph E bijectively correspond to their counterparts in $\beta(E)$, and the same is true for joints. Star articulation degrees of corresponding joints are the same, and so are then the cyclicities of E and $\beta(E)$. The cyclomatic number, however, may change when a hypergraph is blown up, as is evident from the second example given above.

On the other hand, the cyclicity and the cyclomatic number are also similar in certain respects: both depend only on the maximal edges, both decrease on subhypergraphs, and both are additive on compositions. The cyclicity satisfies $\gamma(E) = \gamma(\text{Max}(E))$ by definition, and it is shown in [1] that the cyclomatic number also has this property.

We have seen that the cyclicity decreases on subhypergraphs (Theorem 4). So does the cyclomatic number; since this is not mentioned in [1], it will not hurt to prove it here. Notice that the maximum weight of a subforest of the intersection graph of E is the same as the maximum weight of a tree on the set of nodes E , provided we allow a tree to have arcs of weight 0, which we do.

Theorem 15. *If F is a subhypergraph of E , then $\mu(F) \leq \mu(E)$.*

Proof. We can assume, without loss of generality, that E is a simple hypergraph and that $F := E[U]$, where $U = (\bigcup E) \setminus \{u\}$ for some vertex $u \in \bigcup E$. (It does not matter that $E[U]$ might not be simple.) Write $S := \text{St}_E(u)$. The mapping $S \rightarrow S[U]: s \mapsto s \setminus \{u\}$ is bijective and $S[U]$ has no edge in common with $E(U)$, so $E[U]$ is a disjoint union of $E(U)$ and $S[U]$. Let T be a maximum weight tree for E , i.e., $w(T) = w_E$. Since the mapping $E \rightarrow E[U]: e \mapsto e \setminus \{u\}$ is a bijection, it maps the tree T on nodes E to a tree T' on nodes $E[U]$. We have

$$\mu(E) = \sum_{e \in E} |e| - |\bigcup E| - w(T),$$

$$\mu(E[U]) \leq \sum_{e' \in E[U]} |e'| - |\bigcup E[U]| - w(T').$$

Going from the former to the latter, we lose $|S|$ in the first term (i.e. the sum) and gain 1 in the second. Since the graph induced by the tree T on the set of nodes S is a forest, we gain at most $|S| - 1$ in the third term. All in all we have a net deficit (possibly zero), meaning that $\mu(E[U]) \leq \mu(E)$. \square

Another property shared by the cyclicity and the cyclomatic number is additivity on *compositions of hypergraphs*. Let E_1 and E_2 be hypergraphs with disjoint spans, and let φ be a bijection of a subedge g_1 of E_1 onto a subedge g_2 of E_2 . Given these data, we identify in the hypergraph $E_1 \cup E_2$ each vertex u in g_1 with the vertex $\varphi(u)$ in g_2 , thus obtaining a hypergraph E , the *composition of E_1 and E_2* . Hypergraphs E_1 and E_2 are embedded into E as partial hypergraphs, still denoted E_1 and E_2 , in such a way that $E = E_1 \cup E_2$ and that $(\bigcup E_1) \cap (\bigcup E_2) =: g$ is a subedge of both E_1 and E_2 ; we will observe composition only in this ‘internal’ form, and will say that it is *over g* . If E is a composition of simple hypergraphs E_1 and E_2 over g , then E need not be simple; there are three possibilities:

- (a) g is a proper subedge of both E_1 and E_2 : E_1 and E_2 have no edge in common and E is simple;
- (b) g is an edge of, say, E_1 and is a proper subedge of E_2 : in this case $\text{Max}(E) = (E_1 \setminus \{g\}) \cup E_2$, a disjoint union;
- (c) g is an edge of both E_1 and E_2 ; now E is simple.

We will now prove additivity first for the cyclicity and then for the cyclomatic number of a hypergraph.

Proposition 16. *If a hypergraph E is a composition of hypergraphs E_1 and E_2 , then $\gamma(E) = \gamma(E_1) + \gamma(E_2)$.*

Proof. We may assume that E_1 and E_2 are simple, since this does not affect the generality of the proof. Let g be the common subedge over which E_1 and E_2 are

composed. Take any join-graphs G_1 and G_2 of E_1 and E_2 , respectively, choose an edge $h_1 \ni g$ in E_1 and an edge $h_2 \ni g$ in E_2 , and construct a graph G , as follows. In case (a) (of the three cases (a), (b), and (c) mentioned in the text) add to a graph $G_1 \cup G_2$ the arc $\{h_1, h_2\}$. In case (b) remove from the graph $G_1 \cup G_2$ the node g and reconnect its neighbours to the node h_2 . Finally, in case (c) take for G the union $G_1 \cup G_2$. The cyclomatic number of the graph G is in all three cases the sum of the cyclomatic numbers of G_1 and G_2 . As a consequence of Corollary 11 it suffices to show that G is a join-graph of $\text{Max}(E)$; we will do this only for case (a), since the argument can be easily adapted to the other two cases. We show first that G joins $\text{Max}(E) = E$. Any two edges of E_1 , or of E_2 , are joined in G_1 or in G_2 , respectively, so are also joined in G . Otherwise, if e_1 is an edge of E_1 and e_2 an edge of E_2 , then $e_1 \cap e_2 \subseteq g = h_1 \cap h_2$, hence $e_1 \cap e_2 \subseteq e_1 \cap h_1$ and $e_1 \cap e_2 \subseteq e_2 \cap h_2$; if we now join e_1 to h_1 in G_1 , pass along the arc $\{h_1, h_2\}$ to h_2 , then join h_2 to e_2 in E_2 , we have joined e_1 to e_2 in G . It remains to verify the minimality of G . We cannot remove from G the arc $\{h_1, h_2\}$, because the nodes h_1 and h_2 are not connected, much less joined, in the remaining graph. If a is an arc of, say, G_1 , then its end-nodes are not joined in the graph $G \setminus \{a\}$, since otherwise the shortest path joining the ends of a in $G \setminus \{a\}$ would lie entirely within the graph $G_1 \setminus \{a\}$, which is impossible in view of the minimality of G_1 . \square

Proposition 17. *If a hypergraph E is a composition of hypergraphs E_1 and E_2 , then $\mu(E) = \mu(E_1) + \mu(E_2)$.*

Proof. If E_1 and E_2 are composed over g , we can assume that $E_1 \cap E_2 = \{g\}$, for otherwise we can take $F_1 := \text{Max}(E_1) \cup \{g\}$, $F_2 := \text{Max}(E_2) \cup \{g\}$, and $F := F_1 \cup F_2$ instead of E_1 , E_2 , and E , without affecting the premises or the conclusion of the proposition. Comparing the expression for $\mu(E)$ (given in the introduction) with that for $\mu(E_1) + \mu(E_2)$, we find that they are equal if and only if the weight w_E is the sum of the weights w_{E_1} and w_{E_2} .

Let T_1 and T_2 be maximum weight trees on E_1 and E_2 , respectively. Then $T := T_1 \cup T_2$ is a tree on E , and $w_{E_1} + w_{E_2} = w(T_1) + w(T_2) = w(T) \leq w_E$.

Conversely, let T be a maximum weight tree on E . Removing (for a moment) the node g from the tree T yields a forest of subtrees S_1, \dots, S_n rooted at the neighbours of the node g in the tree T . If each subtree S_j has all nodes either in $E_1 \setminus \{g\}$ or in $E_2 \setminus \{g\}$, then clearly $T = T_1 \cup T_2$, where T_1 is a tree on E_1 and T_2 is a tree on E_2 , and we have the opposite inequality $w_{E_1} + w_{E_2} \geq w(T_1) + w(T_2) = w(T) = w_E$. Otherwise, some subtree S_j contains an arc $\{e_1, e_2\}$, with the node e_2 farther from the root of S_j than the node e_1 , which crosses, say, from $e_1 \in E_1 \setminus \{g\}$ to $e_2 \in E_2 \setminus \{g\}$. Since $e_1 \cap e_2 \subseteq g$, we have $e_1 \cap e_2 \subseteq g \cap e_2$. Remove from the tree T the arc $\{e_1, e_2\}$, then add the arc $\{g, e_2\}$; the result is again a tree on E , which has one crossing less than T . The weight could have only increased, hence has remained the same. \square

8. Conclusions

We introduced and examined the acyclicity of a hypergraph, an extension of the cyclomatic number of a graph. We associated with a hypergraph circulant graphs and join-graphs, which have the cyclomatic number equal to the cyclicity of the hypergraph. From the structure theorem and the construction theorem for join-graphs we gained an insight into the meaning of the summands in the formula for cyclicity.

We have seen two extensions of the cyclomatic number of a graph to hypergraphs. Let us show how to construct other similar ‘cyclicity measures’, i.e., functions from hypergraphs to non-negative integers which for graphs give the usual cyclomatic number and are zero precisely on acyclic hypergraphs. Let α map hypergraphs to non-negative integers, giving zero on acyclic hypergraphs, and define another function $\Sigma\alpha$ from hypergraphs to non-negative integers by

$$\Sigma\alpha(E) := \sum_{u \in \bigcup E} \alpha(\text{St}_E(u)).$$

Then $\Sigma\alpha(E)$ is zero on every hypergraph E in which all stars $\text{St}_E(u)$ of vertices $u \in \bigcup E$ are acyclic; in particular, $\Sigma\alpha$ is zero on graphs and acyclic hypergraphs. Now, if β is a cyclicity measure, then $\beta + k \cdot \Sigma\alpha$ is again a cyclicity measure, for any non-negative integer k .

To give an example, consider $\Sigma\gamma$. We have

$$\Sigma\gamma(E) = \sum_f (\delta_E(f) - 1) \cdot |f| + |\bigcup E|,$$

where f runs through all joints and maximal edges, and possibly some other subedges. A hypergraphs E has $\Sigma\gamma(E) = 0$ precisely when all of its vertices have acyclic stars. Using $\Sigma\gamma$, we obtain two infinite sequences of cyclicity measures $\gamma + k \cdot \Sigma\gamma$ and $\mu + k \cdot \Sigma\gamma$, for $k = 0, 1, 2, \dots$. An interesting one is

$$(\mu + \Sigma\gamma)(E) = \sum_{f \in \Psi(E)} (\delta_E(f) - 1) \cdot |f| - w_E.$$

For a simple hypergraph E the first part of the right-hand side, the sum, is the weight of any join-graph of E . The fact that the whole expression is always non-negative leads us to enquire whether every maximum weight tree is a part of some join-graph. This is indeed so. It can be shown, by a reasoning similar to that in the proof of the structure theorem for join-graphs, that every arc of a maximum weight tree is an articulated pair, and that for every joint f the set of all arcs $\{e_1, e_2\}$ of the tree with $e_1 \cap e_2 = f$ is a part of a tree set over f . The construction theorem then gives the rest.

A possible future direction of research on cyclicity measures would be to study classes of cyclicity measures satisfying additional conditions, say some stronger form of additivity. One of the aims could be to discover simple conditions determining a unique cyclicity measure.

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