A triad of cubic curves associated with a triangle

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Lemma 1. *Let $\mathscr{C}$ be a circle with center O, let A be a point on the circle, let P be a point in the plane of the circle but not on the circle, and let q be the line through the point A perpendicular to the line segment $\overline{AP}$. Reflect the point P on the circle's center O, obtaining the point P'. If the point P is inside resp. outside the circle $\mathscr{C}$, then the point P' is on the same resp. opposite side of the line q than the point P.*

Proof. Let $X \mapsto X'$ be the reflection through the circle's center O. Figure 1 tells the proof without words. □

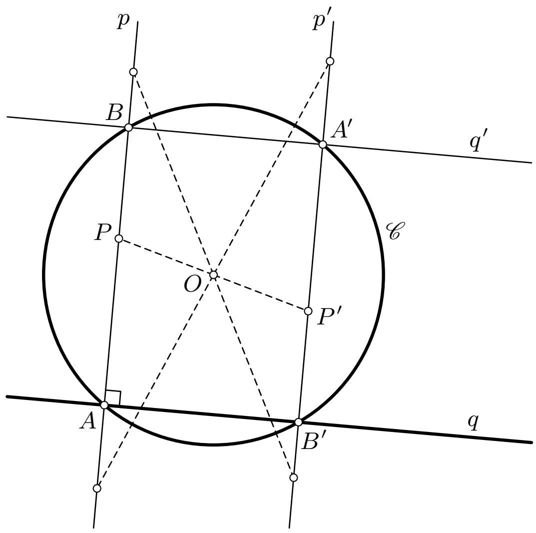

Figure 1. Behold!

Let Δ be a triangle $A_1A_2A_3$, and let E be any point in the plane of the triangle. The feet E_1, E_2, E_3 of the perpendiculars through the point E to, respectively, the sidelines $q_1 = \overleftrightarrow{A_2A_3}$, $q_2 := \overleftrightarrow{A_3A_1}$, $q_3 := \overleftrightarrow{A_1A_2}$, are the vertices of the *pedal triangle* $E_1E_2E_3$, denoted by $\Pi(E)$ $\big(= \Pi_\Delta(E)\big)$, of the point E with respect to the triangle Δ. If the point E is not a vertex of the triangle Δ, then the three pedal points E_1, E_2, E_3 are different from each other. In more detail, let (i, j, k) be a permutation of $(1, 2, 3)$; we have $E_i = E_j$ iff $E = E_i = E_j = A_k$, and in such a case the third pedal point E_k, the foot of the altitude dropped from the vertex A_k of the triangle Δ to its opposite sideline q_k, is different from the two equal pedal points $E_i = E_j$.

We shall denote by $\mathscr{O}$ the circumcircle of the triangle Δ, by R its circumradius, by $\mathscr{C}(E)$ the circumcircle of the pedal triangle $\Pi(E)$ (see Figure 2), by $\mathit{\Delta}$ the (oriented, that is, signed) area of the triangle Δ, and by $\mathit{\Pi}(E)$ the area of the pedal triangle $\Pi(E)$.

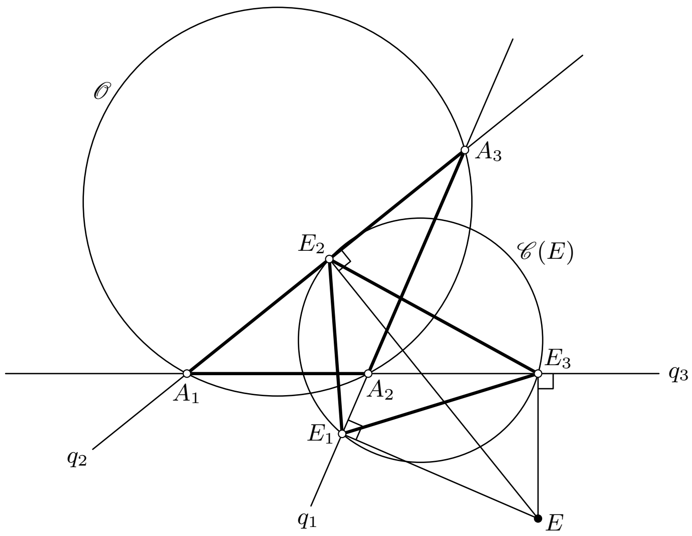

Figure 2

If $\mathscr{K}$ is a circle with center Q and of radius r, and X is any point in the plane of the circle, then we denote by $\mathcal{P}_{\mathscr{K}}(X)$ the power of the point X with respect to the circle $\mathscr{K}$:

$$\mathcal{P}_{\mathscr{K}}(X) := |QX|^2 - r^2 .$$

Proposition 2. *The area of the pedal triangle* $\Pi(E)$ *is proportional to the power of the point* E *with respect to the circumcircle* $\mathscr{O}$ *of the triangle* Δ*, with the proportionality factor depending only on the shape and the orientation of the triangle* Δ*:*[1]

$$\mathit{\Pi}(E) = -\frac{\mathit{\Delta}}{4R^2} \cdot \mathcal{P}_{\mathscr{O}}(E) . \tag{1}$$

The identity (1) is easily verified by Mathematica, since both sides are rational functions of the coordinates of the vertices of the triangle Δ and of the point E, all coordinates taken in some Cartesian coordinate system. The verification is very fast (and can be done, with some patience and care, by hand), if the coordinate system has the origin at a vertex of a triangle Δ and has another triangle's vertex on its first axis.

Henceforth we assume that the triangle Δ is positively oriented. Formula (1) for the area $\mathit{\Pi}(E)$ of the pedal triangle tells us that the pedal triangle is positively oriented when the point E is inside the circle $\mathscr{O}$, and that it is negatively oriented when the point E is outside the circle $\mathscr{O}$ (which is the case in Figure 2).

Suppose that the point E lies on the circumcircle $\mathscr{O}$. In this case $\mathit{\Pi}(E) = 0$, thus the pedal points E_1, E_2, E_3 are collinear; since never $E_1 = E_2 = E_3$, the pedal points determine a line, know as the *Simson line* (though it should be Wallace's line). Now suppose, in addition, that E is not a vertex of Δ; then the collinear pedal points E_1, E_2, E_3 are all different, and the Simson line is the degenerate circumcircle $\mathscr{C}(E)$ of the pedal triangle $\Pi(E)$, whose center is the point F at infinity in the direction orthogonal to the Simson line. The point F is also the isogonal conjugate of the point E

[1] The proportionality factor is $-\mathit{\Delta}/4R^2 = -\frac{1}{2}\,\mathrm{sgn}(\Delta)\sin\alpha_1\sin\alpha_2\sin\alpha_3$, where each α_k is the interior angle of the triangle Δ at its vertex A_k, and $\mathrm{sgn}(\Delta) = \mathrm{sgn}(\mathit{\Delta}) = \pm 1$ is the orientation of the triangle Δ.

with respect to the triangle Δ, and so the following holds (in the limit, as a point E not on the circumcircle $\mathscr{O}$ approaches a position on it): for each k ($=$ 1, 2, 3), the line through the vertex A_k perpendicular to the Simson line, and the line through the pedal point E_k parallel to the sideline q_k, meet in the point E'_k on the circumcircle $\mathscr{O}$ (see Figure 3). In case you feel uneasy with the 'in the limit' argument, here is a direct

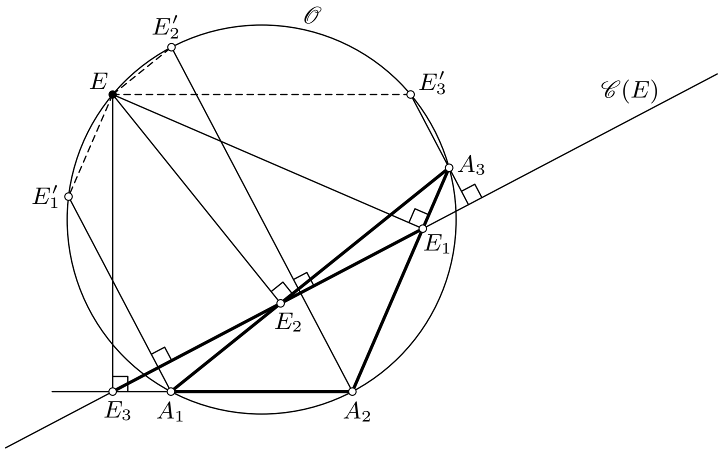

Figure 3

proof (Figure 4). Let E be any point in the plane of the triangle Δ, let E_1 and E_2 be

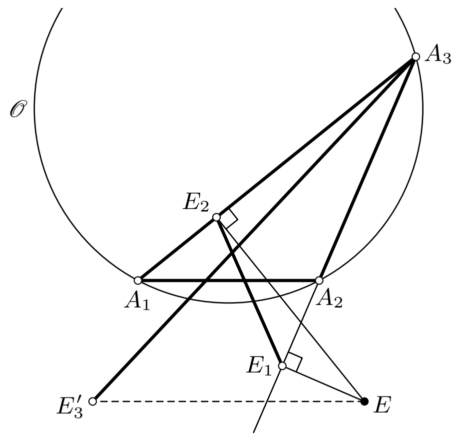

Figure 4

the first two pedal points of E with respect to Δ, and let the point E'_3 be obtained by reflecting the point E through the perpendicular bisector of the side $\overline{A_1A_2}$ of Δ; then

$$(E_2 - E_1) \cdot (E'_3 - A_3) \;=\; \frac{4\,\varDelta^2\,(a_1^2 - a_2^2)}{a_1^2\,a_2^2\,a_3^2} \cdot \mathcal{P}_{\mathscr{O}}(E)\;. \tag{2}$$

where a_1, a_2, a_3 are the lengths of the sides of the triangle Δ. The identity (2) can be easily verified by MATHEMATICA, just like the identity (1), and for the same reason. Now, if the point E is on the circumcircle $\mathscr{O}$, then E'_3 is the other intersection point of $\mathscr{O}$ and the line through E parallel to the sideline $q_3 = \overleftrightarrow{A_1A_2}$, and because of $\mathcal{P}_{\mathscr{O}}(E) = 0$ the identity (2) tells us that the line segment $\overline{A_3E'_3}$ is orthogonal to the Simson line $\overleftrightarrow{E_1E_2}$ (which contains also the third pedal point E_3).

Let E be a point in the plane of the triangle Δ that does not lie on the circumcircle $\mathscr{O}$ of the triangle. Any such point E has (associated with it) a nondegenerate pedal triangle $\Pi(E)$ with respect to Δ, hence has a circumcircle $\mathscr{C}(E)$ of the pedal triangle (which is a true circle, not a line), and thus has the isogonal conjugate F with respect to Δ which is the point obtained by reflecting the point E through the center $Q(E)$ of the pedal circumcircle $\mathscr{C}(E)$. The following is a consequence of Lemma 1:

Corollary 3. *Let E be a point in the plane of the triangle Δ not on the circumcircle of the triangle. If the point E is inside/outside its pedal circumcircle $\mathscr{C}(E)$, then its isogonal conjugate F is on the same/opposite side of each of the three sidelines of the triangle Δ.*

In particular, if E is in the interior of the triangle Δ, then so is F; this is because in this case E lies in the interior of its pedal triangle $\Pi(E)$, and this interior is contained in the interior of the pedal circumcircle $\mathscr{C}(E)$.

We know that the point E and its isogonal conjugate F are the foci of the conic inscribed into the triangle[2] which is uniquely determined by its focus E (or by its focus F), and that the circumcircle $\mathscr{C} = \mathscr{C}(E)$ of the pedal triangle $\Pi(E)$, which is also the circumcircle of the pedal triangle $\Pi(F)$, is the auxiliary circle of this conic. The inscribed conic is either an ellipse or a hyperbola: it is an ellipse if E is inside $\mathscr{C}(E)$ and it is a hyperbola if E is outside $\mathscr{C}(E)$.[3] In Figure 5 we have the inscribed ellipse

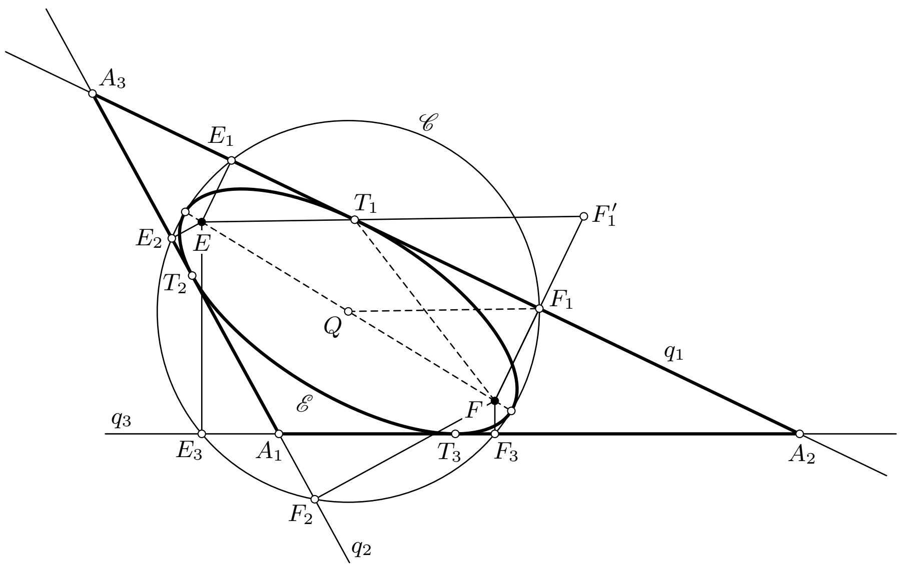

Figure 5

in the case where E, and hence F, is in the interior of Δ; in the figure we can also see a construction of the point of tangency of the inellipse on one of the sides of the triangle. (Figure 5 is a rip-off of Fig. 1 in J. Steiner's paper [1].)

[2]The conic is actually inscribed, just as incircle is inscribed, only when E and F are in the interior of the triangle Δ; otherwise one would prefer to say that it is *ex*scribed. Alas, this is the accepted terminology, the conic is said to be inscribed in all cases; it is also called an *inconic* for short.

[3] It is a parabola if the point E lies on the circumcircle $\mathscr{O}$ of the triangle Δ (but is not a vertex of the triangle), which we excluded from the current discussion; we shall come to it later. Mark that with E on $\mathscr{O}$ the pedal circumcircle $\mathscr{C}(E)$ is a line and we are unable to tell which side of this line is the inside (or the outside) of the circumcircle.

We want to know the exact parts of the plane, minus the circumcircle $\mathscr{O}$ of the triangle Δ, in which the point E is inside/outside its pedal circumcircle $\mathscr{C}(E)$; well, this happens when the power of E with respect to $\mathscr{C}(E)$ is strictly negative/positive, but where is that? The following identity will help us answer this question. Recall that $\mathcal{P}_{\mathscr{K}}(X)$ denotes the power of a point X with respect to a circle $\mathscr{K}$, and that we denoted the (oriented) area of the triangle Δ by $\mathit{\Delta}$ and the area of the pedal triangle $\Pi(E)$ by $\mathit{\Pi}(E)$. Moreover, for (i,j,k) a cyclic permutation of $(1,2,3)$, we denote the (oriented) area of the triangle A_iA_jE by $\mathit{\Delta}_k(E)$.

Proposition 4. *If E is a point in the plane of the triangle Δ not on its circumcircle $\mathscr{O}$, then*

$$\mathcal{P}_{\mathscr{C}(E)}(E)\cdot\mathcal{P}_{\mathscr{O}}(E) \;=\; 4\,\frac{\mathit{\Delta}_1(E)\,\mathit{\Delta}_2(E)\,\mathit{\Delta}_3(E)}{\mathit{\Delta}}\;. \tag{3}$$

(The identity (3) can be verified by MATHEMATICA, as were two previous identities.) Since E does not lie on $\mathscr{O}$, the power $\mathcal{P}_{\mathscr{O}}(E)$ is nonzero. Since E has to lie either inside or outside its pedal circumcircle $\mathscr{C}(E)$ (which is a true circle), but not on $\mathscr{C}(E)$ itself, the power $\mathcal{P}_{\mathscr{C}(E)}(E)$ must be nonzero. Therefore the right side of the identity (3) must be nonzero. The area $\mathit{\Delta}_k(E)$ is nonzero iff the point E does not lie on the sideline $q_k = \overleftrightarrow{A_iA_j}$ of the triangle Δ; so we must remove from the plane, besides the circumcircle of Δ, also the three sidelines of Δ. The remaining region (an open subset of the plane) has ten connected components. The sign of the power $\mathcal{P}_{\mathscr{C}(E)}(E)$ is constant on each of the ten subregions; a moment's thought shows that the signs on the subregions are as in Figure 6.

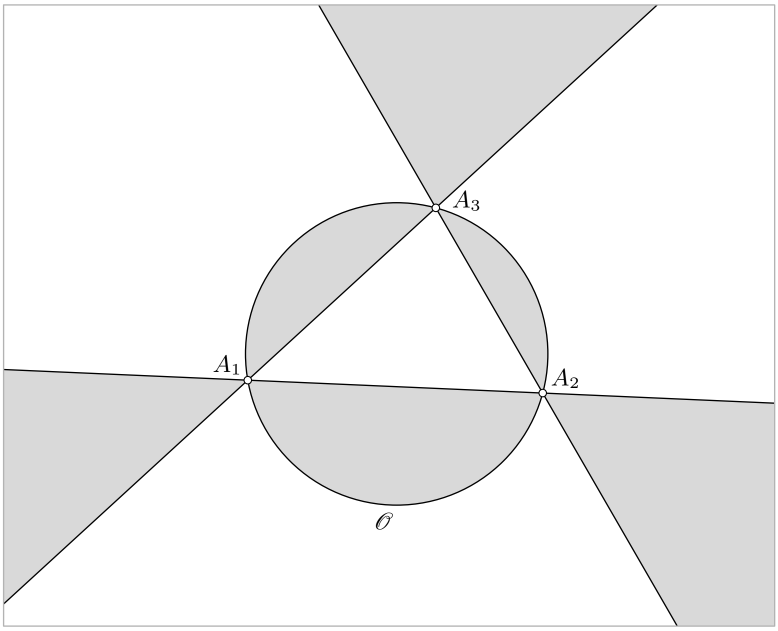

Figure 6. The sign of $\mathcal{P}_{\mathscr{C}(E)}(E)$: white is strictly negative, gray strictly positive.

Indeed, the fraction on the right hand side of (3) can be rewritten as

$$\frac{\mathit{\Delta}_1(E)}{\mathit{\Delta}}\cdot\frac{\mathit{\Delta}_2(E)}{\mathit{\Delta}}\cdot\frac{\mathit{\Delta}_3(E)}{\mathit{\Delta}}\cdot\mathit{\Delta}^2 \;=\; x_1\cdot x_2\cdot x_3\cdot\mathit{\Delta}^2\;,$$

where $\mathit{\Delta}_k/\mathit{\Delta} = x_k$, the k-th affine coordinate of the point E with respect to the triangle Δ, has a positive sign if E is in the open half-plane bounded by q_k that contains A_k, and has a negative sign if E is in the other open half-plane. From this it is easy to figure out that the sign of $\mathcal{P}_{\mathscr{C}(E)}(E)$ behaves as shown in Figure 6.

An immediate consequence is that the isogonal conjugation is an involution on each of the white subregions, while it bijectively maps the interior of each of the gray segments of the circumcircle $\mathscr{O}$ onto the interior of the gray angle emanating outwards from the vertex of the triangle Δ opposite to the segment, and conversely (and inversely).

Now we turn to the question where does the inconic associated with a point E touch the sidelines of the triangle Δ. Here we need to explain the "where" in the question. Each triangle's sideline q_k is cut by the vertices A_i and A_j into three parts, the triangle's side $\overline{A_iA_j}$ opposite the vertex A_k (a line segment), and the two half-lines starting at the two vertices and pointing away from the side; we want to know within which of these three parts of the sideline q_k (if any) lies its point of tangency with the inconic.

If the point E is in the interior of the triangle Δ, which is the central white subregion in Figure 6, then the associated inconic is an ellipse which is (truly) inscribed in the triangle, like the ellipse $\mathscr{E}$ of Figure 5; it touches each sideline q_k within the side $\overline{A_iA_j}$.

If the point E lies in one of the other three white subregions in Figure 6, say in the one opposite the vertex A_2 as in Figure 7, then the associated inconic is an ellipse $\mathscr{E}$ which

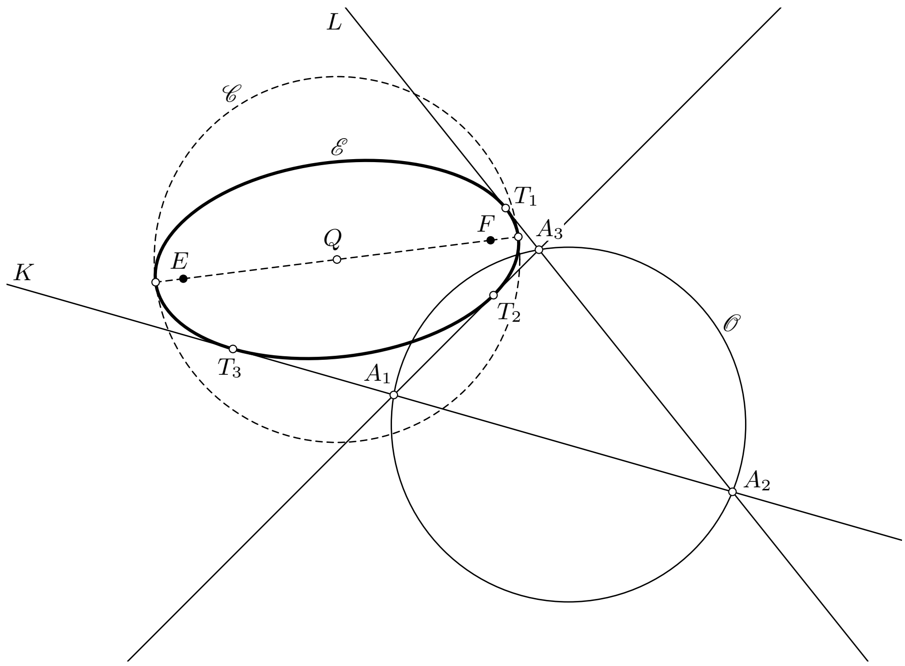

Figure 7

is exscribed[4] to the triangle Δ, and which touches the sideline q_1 within the half-line $\overrightarrow{A_3L}$, the sideline q_2 within the side $\overline{A_3A_1}$, and the sideline q_3 within the half-line $\overrightarrow{A_1K}$.

In each of the foregoing two cases we therefore know the answer to our question, and the answer is not changing while the point E is roaming around any white subregion.

The situation is different when the point E is in a gray subregion. In this case the inconic is a hyperbola exscribed to the triangle Δ. For instance, if E is in the gray segment of the circumcircle $\mathscr{O}$ cut away by the sideline q_1, then its isogonal conjugate

[4] Sorry, cannot make myself say "inscribed", 'cause it isn't.

is in the gray angle emanating outwards from the vertex A_1. This is the situation in Figure 8. The branch $\mathscr{H}_E$ of the exhyperbola $\mathscr{H}$, the one on the side of its focus E,

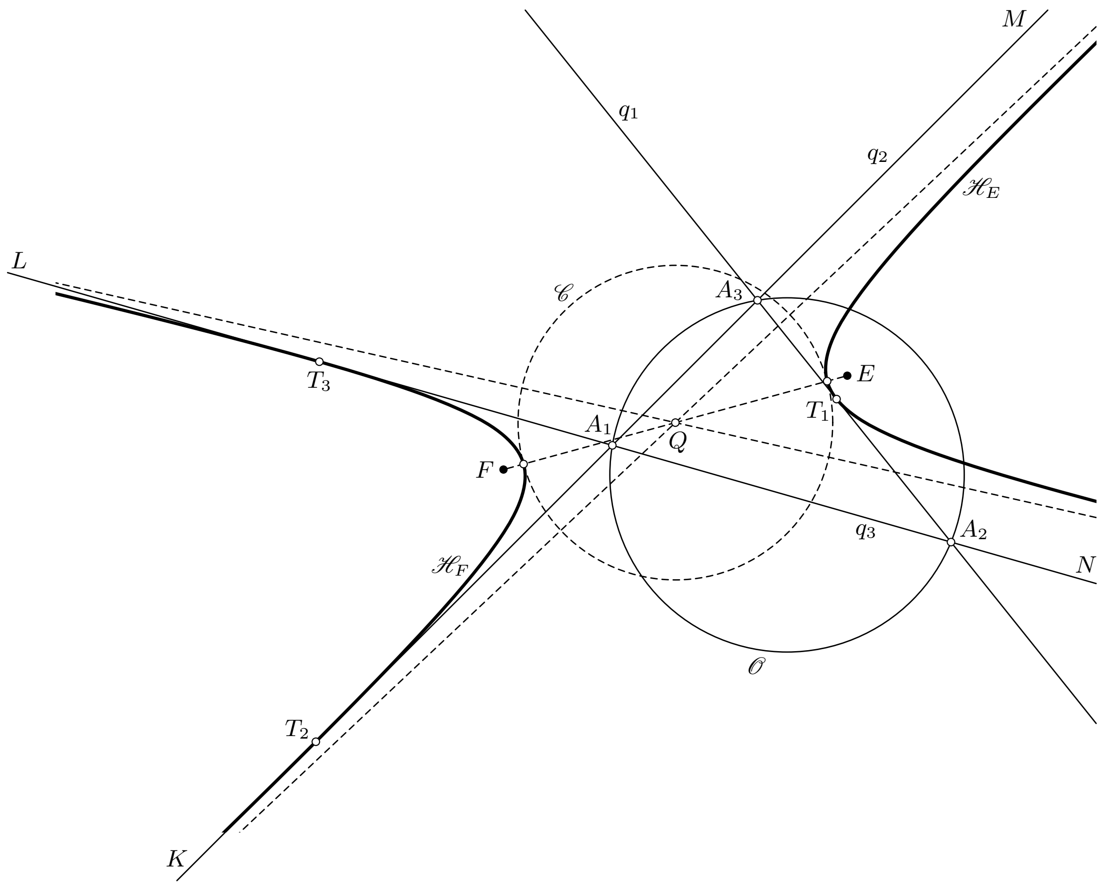

Figure 8

is contained in the truncated angle MA_3A_2N, while the other branch $\mathscr{H}_F$ is contained in the angle KA_1L.[5] The branch $\mathscr{H}_E$ touches the sideline q_1 within the side $\overline{A_2A_3}$: $\mathscr{H}_E$ touches q_1 because the pedal circumcenter $Q = Q(E)$ of E (that is, the center of the circumcircle of the pedal triangle of E) lies on the same side of the sideline q_1 as the vertex A_1 and hence lies on the opposite side of q_1 to the hyperbola's focus E, and furthermore, $\mathscr{H}_E$ touches q_1 within $\overline{A_2A_3}$ because it cannot touch it anywhere else, contained in the truncated angle MA_3A_2N as it is. Likewise the branch $\mathscr{H}_F$ touches the sideline q_2 within the half-line $\overrightarrow{A_1K}$ because Q lies on the same side of the sideline q_2 as the vertex A_2 and hence lies on the opposite side of q_2 to the focus F, and it touches the sideline q_3 within the half-line $\overrightarrow{A_1L}$ because Q lies on the same side of the sideline q_3 as the vertex A_3 and hence lies on the opposite side of q_3 to the focus F. In short, the exhyperbola $\mathscr{H}$ touches the sidelines of the triangle Δ where it does because all three affine coordinates of the pedal circumcenter Q with respect to the triangle Δ are strictly positive.

So the next thing we will want to know, for each $k = 1, 2, 3$, are the sets of all points E in the plane, minus the circumcircle $\mathscr{O}$ of Δ, for which the k-th affine coordinate

[5] Mind that an inconic cannot cross any of the sidelines, since each of the three sidelines is its tangent (perhaps with the point of tangency at infinity, meaning that it is an asymptote of the inconic).

of the pedal circumcenter $Q(E)$ with respect to the triangle Δ is strictly positive/strictly negative/zero. Let $(z_1, z_2, z_3) = [Q(E)]_\Delta$ be the affine coordinates of $Q(E)$. Suppose that E lies in the interior of the segment $\mathscr{S}_k$ of the circumcircle $\mathscr{O}$ cut from it by the sideline q_k, the one on the opposite side of q_k to the vertex A_k. We shall soon see that in this case $z_k = z_k(E)$ is strictly positive, so that $Q(E)$ lies on the opposite side to E, because of which it is the hyperbola's branch $\mathscr{H}_E$ that touches the sideline q_k, and it does that within the side $\overline{A_iA_j}$. The isogonal conjugate F of E lies in the interior of the angle $\mathscr{A}_k$ vertically opposite to the internal angle of the triangle Δ at its vertex A_k. Let i be one of the two indices 1, 2, 3 which are different from k. If $z_i > 0$ then $Q(E)$ lies on the opposite side of the sideline q_i to F, thus the hyperbola's branch $\mathscr{H}_F$ touches q_i, and it touches it within the half-line which is one of the legs of the angle $\mathscr{A}_k$. If $z_i < 0$ then $Q(E)$ lies on the opposite side of the sideline q_i to E and now it is the branch $\mathscr{H}_E$ that touches q_i within the half-line that is one of the legs of the truncated angle which contains $\mathscr{H}_E$. Finally, if $z_i = 0$, then the hyperbola $\mathscr{H}$ touches the sideline q_i at infinity, that is, the sideline q_i is an asymptote of $\mathscr{H}$.

Let the point E have the barycentric coordinates $(E)_\Delta = (x_1 : x_2 : x_3)$ with respect to the triangle Δ, and let a_1, a_2, a_3 be the lengths of the sides of Δ. The isogonal conjugate F of E has the barycentric coordinates $(F)_\Delta = \big(a_1^2/x_1 : a_2^2/x_2 : a_3^2/x_3\big)$, and so the pedal circumcenter $Q(E)$ of E has the affine coordinates

$$\begin{aligned}
\big[Q(E)\big]_\Delta &= \frac{1}{2}\left(\frac{(x_1, x_2, x_3)}{x_1+x_2+x_3} + \frac{\big(a_1^2/x_1,\, a_2^2/x_2,\, a_3^2/x_3\big)}{a_1^2/x_1 + a_2^2/x_2 + a_3^2/x_3}\right) \\
&= \frac{\Big(f_1(x_1,x_2,x_3),\, f_2(x_1,x_2,x_3),\, f_3(x_1,x_2,x_3)\Big)}{p(x_1,x_2,x_3)}\ ,
\end{aligned}$$

where

$$f_1(x_1,x_2,x_3) = x_1^2\big(a_3^2x_2 + a_2^2x_3\big) + a_1^2\big(2x_1 + x_2 + x_3\big)x_2x_3\ ,$$

where $f_2(x_1,x_2,x_3)$ is obtained from $f_1(x_1,x_2,x_3)$ by rotating the indices $1 \mapsto 2 \mapsto 3 \mapsto 1$ in x_1, x_2, x_3 as well as in a_1, a_2, a_3, where $f_3(x_1,x_2,x_3)$ is obtained from $f_2(x_1,x_2,x_3)$ by rotating the indices again, and where

$$p(x_1,x_2,x_3) = 2\big(x_1+x_2+x_3\big)\big(a_3^2x_1x_2 + a_2^2x_1x_3 + a_1^2x_2x_3\big)\ .$$

Suppose that x_1, x_2, x_3 are the affine coordinates of E with respect to Δ. Then we have $x_1 + x_2 + x_3 = 1$, whence

$$\begin{aligned}
f_1(x_1,x_2,x_3) &= f_1(x_1,\, x_2,\, 1-x_1-x_2) \\
&= -a_1^2(1+x_1)\,x_2^2 + \big(a_1^2 - (a_1^2+a_2^2-a_3^2)x_1^2\big)\,x_2 + a_2^2(1-x_1)x_1^2\ , \\
p(x_1,x_2,x_3) &= -2\mathcal{P}_{\mathscr{O}}(E)\ .
\end{aligned}$$

Solving $f_1(x_1,\, x_2,\, 1-x_1-x_2) = 0$ for x_2, we obtain, for $\sigma = \pm 1$,

$$x_2 = g_2^{(\sigma)}(x_1) := \frac{a_1^2 - (a_1^2+a_2^2-a_3^2)x_1^2 + \sigma\sqrt{D_1}}{2a_1^2(1+x_1)}\ ,$$

where

$$D_1 = \Big(a_1^2 - \big(a_1^2 - (a_2-a_3)^2\big)x_1^2\Big)\Big(a_1^2 + \big((a_2+a_3)^2 - a_1^2\big)x_1^2\Big)\ .$$

Suppose that $a_2 \neq a_3$. The two branches $g_2^-(x_1) = g_2^{(-1)}(x_1)$ and $g_2^+(x_1) = g_2^{(+1)}(x_1)$ of the two-valued function $g_2(x_1)$ are defined on the interval $-d_1 \leqslant x_1 \leqslant d_1$, where

$$d_1 \;=\; \frac{a_1}{\sqrt{a_1^2-(a_2-a_3)^2}} \;. \tag{4}$$

The two branches of $g_2(x_1)$ are smoothly glued together at $x_1 = -d_1$ and $x_1 = d_1$, where we have, for $\tau = \pm 1$,

$$g_2(\tau d_1) \;=\; g_2^-(\tau d_1) \;=\; g_2^+(\tau d_1) \;=\; \frac{a_2(a_2-a_3)}{(a_2-a_3)^2 - a_1^2 - \tau a_1\sqrt{a_1^2-(a_2-a_3)^2}} \;;$$

the denominator, which can be rewritten as $-\sqrt{a_1^2-(a_2-a_3)^2}\big(\sqrt{a_1^2-(a_2-a_3)^2}+\tau a_1\big)$, is nonzero. The values of $g_2(x_1)$ at $x_1 = 0$ are $g_2^-(0) = 0$ and $g_2^+(0) = 1$, and the first derivatives of $g_2(x_1)$ at $x_1 = 0$ are $(g_2^-)'(0) = 0$ and $(g_2^+)'(0) = -1$. The values of $g_2(x_1)$ at $x_1 = 1$ are 0 and $(a_3^2-a_2^2)/2a_1^2$, but which value is which depends on the ordering of a_2 and a_3: if $a_2 < a_3$ then $g_2^-(1) = 0$ and $g_2^+(1) = (a_3^2-a_2^2)/2a_1^2$, and if $a_2 > a_3$ then $g_1^-(1) = (a_3^2-a_2^2)/2a_1^2$ and $g_2^+(1) = 0$.

The denominator of the expression for $g_2^{(\sigma)}(x_1)$ vanishes at $x_1 = -1$: one of the branches of $g_2(x_1)$ has a simple pole at $x_1 = -1$, while the other branch is smooth at this point. Which branch is which? Once more this depends on the ordering of a_2 and a_3. Suppose that $a_2 < a_3$. The numerator of the expression for $g_2^{(\sigma)}(x_1)$ at $x_1 = -1$ is then

$$a_3^2 \;-\; a_2^2 \;+\; \sigma\cdot|a_2-a_3|\cdot(a_2+a_3) \;=\; (1+\sigma)(a_3^2-a_2^2) \;,$$

which is nonzero if $\sigma = 1$ and is zero if $\sigma = -1$. In this case it is the branch $g_2^+(x_1)$ which has a pole at $x_1 = -1$, while the expression for $g_2^-(x_1)$ can be rewritten as

$$g_2^-(x_1) \;=\; \frac{2a_2^2(x_1-1)x_1^2}{a_1^2-(a_1^2+a_2^2-a_3^2)x^2+\sqrt{D_1}} \;,$$

where the denominator at $x_1 = -1$ is $2(a_3^2-a_2^2) \neq 0$.

We conclude the discussion of the two-valued function $g_2(x_1)$ by considering the special case $a_2 = a_3$, where we have

$$D_1 \;=\; a_1^2\big(1-x_1^2\big)\big(a_1^2+(4a_2^2-a_1^2)x_1^2\big) \;,$$

$$g_2^{(\sigma)}(x_1) \;=\; \frac{1}{2}\big(1-x_1\big) \;+\; \frac{\sigma}{2}\sqrt{\frac{1-x_1}{1+x_1}\cdot\Big(1+\Big(4\Big(\frac{a_2}{a_1}\Big)^2-1\Big)x_1^2\Big)}$$

The function $g_2(x_1)$ has both branches defined on the interval $-1 < x_1 \leqslant 1$, and at $x_1 = -1$ it has a 'fractional pole' of order $\frac{1}{2}$. We still have $g_2^-(0) = 0$ and $g_2^+(0) = 1$, and $(g_2^-)'(0) = 0$ and $(g_2^+)'(0) = -1$. At $x_1 = 1$ the two branches are smoothly glued together at the value $g_2^-(1) = g_2^+(1) = 0$.

The expressions for the functions $g_3^{(\sigma)}(x_2)$ and $g_1^{(\sigma)}(x_3)$ are of course obtained by rotating the indices.

For quite a while we are going to use for concrete examples the triangle Δ with the vertices $A_1 = (0,0)$, $A_2 = (7,-2)$, $A_3 = (3,3)$, whose sides have lengths $a_1 = \sqrt{41}$, $a_2 = 3\sqrt{2}$, $a_3 = \sqrt{53}$.

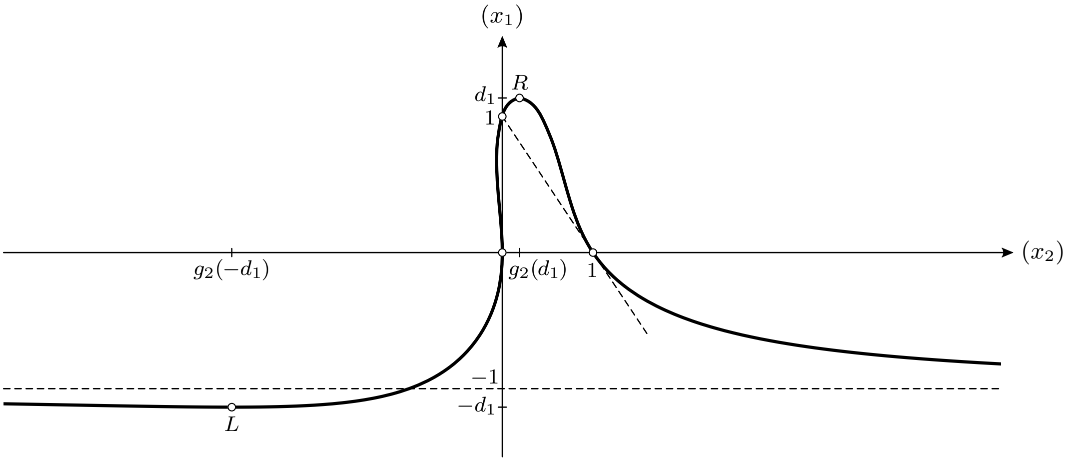

Figure 9. The graph of $x_2 = g_2(x_1)$ $\big($with $g_2(x_1)$ two-valued, except at $x_1 = \pm d_1\big)$.

The graph of the two-valued function $g_2(x_1)$ associated with our staple example triangle Δ is in Figure 9; we flipped the graph onto its side to conserve the vertical space on the page. The graph's arc from L to R is the graph of $g_2^-(x_1)$, the rest of the graph is the graph of $g_2^+(x_1)$. The set of all points E with the zero first affine coordinate z_1 of its pedal circumcenter $Q(E)$ (with respect to Δ) is the curve $\mathscr{K}_1$; see Figure 10. The curve $\mathscr{K}_1$ has a parameterization $x_1A_1 + g_2(x_1)A_2 + \big(1 - x_1 - g_2(x_1)\big)A_3$

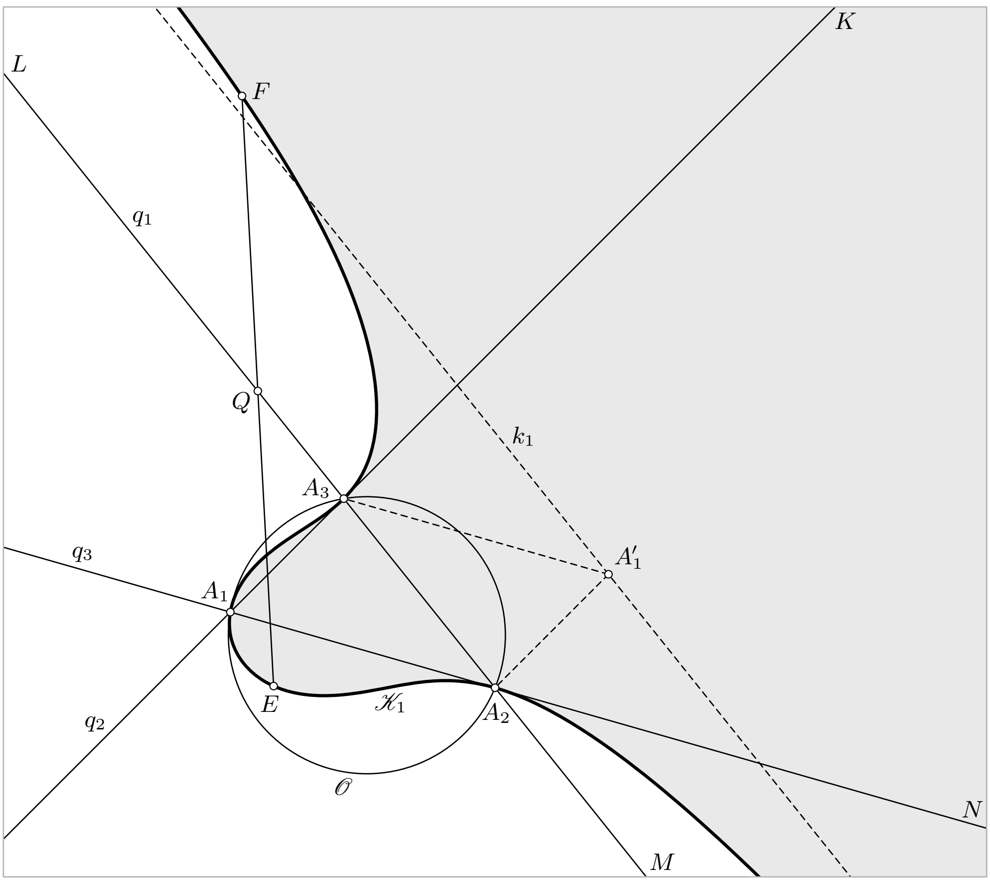

Figure 10

with $-d_1 \leqslant x_1 \leqslant d_1$; the function $g_2(x_1)$ is two-valued, where one of the branches has a simple pole at $x_1 = -1$ which corresponds to the asymptote of $\mathscr{K}_1$ (the line k_1 in Figure 10). The curve $\mathscr{K}_1$ is self-isogonal-conjugate, since any point E on it and its isogonal conjugate F share the same pedal circumcenter Q which lies on the sideline q_1.

The point A_1' has the affine coordinates $(-1,1,1)$ and so the value of $f_1(x_1,x_2,x_3)$ at this point is is $f_1(-1,1,1) = a_2^2 + a_3^2 > 0$. Since the sign of $f_1(E)$ changes only when the point E crosses the curve $\mathscr{K}_1$, the function f_1 is strictly positive on the side of $\mathscr{K}_1$ which contains the point A_1' and is strictly negative on its other side. In Figure 10 the region where f_1 is positive is gray, while the region where f_1 is negative is white.

If the point E is in one of the four white subregions in Figure 6, then it is a focus of an inellipse (inscribed or exscribed) of the triangle Δ, with the center of the inellipse lying either in the interior of the triangle Δ or in the interior of one of the three truncated angles; more precisely, the center of the inellipse lies in the open gray region in Figure 11 (the points N_1, N_2, N_3 are the midpoints of the sides of the triangle Δ), and conversely,

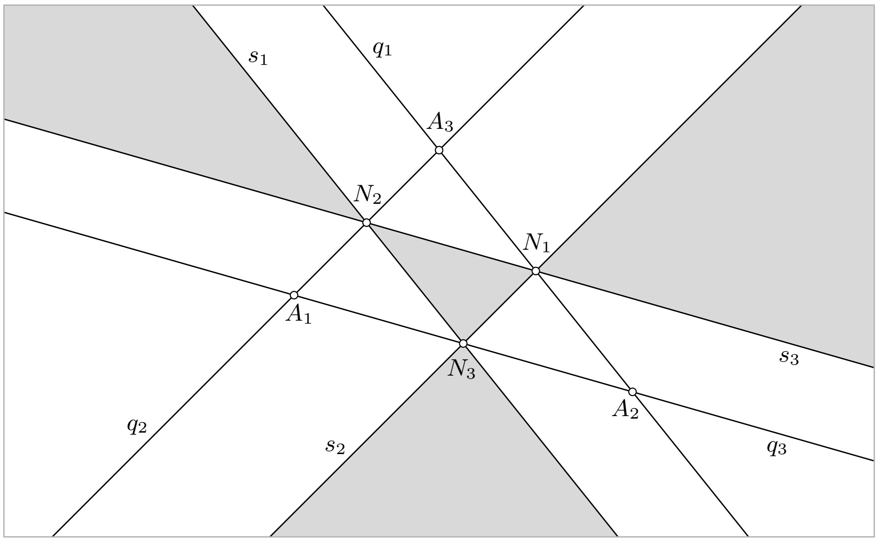

Figure 11

every point in this region is the center of a unique inellipse. In particular this implies that if the point E is in the white region in Figure 6, then the associated pedal circumcenter, which is the center of the inellipse with a focus E, does not lie on any of the sidelines of the triangle Δ. Consequently, if the point E (which avoids the circumcircle $\mathscr{O}$ of the triangle Δ as well as the sidelines of this triangle) has its pedal circumcenter lying on a sideline of the triangle Δ, then E lies in one of the six gray subregions in Figure 6. More precisely, if the pedal circumcenter $Q(E)$ lies on the sideline q_k, then the point E lies in one of the following four subregions: in the interior of one of the two segments of the circumcircle $\mathscr{O}$ of Δ cut away by the sidelines q_i and q_j, or in the interior of one of the angles emanating outwards from the opposite vertices A_i and A_j, which are by isogonal conjugation coupled with the two segments of $\mathscr{O}$. In Figure10 we can see the curve $\mathscr{K}_1$ passing through these four subregions for $k = 1$ and $i = 2$, $j = 3$: the interior of the segment of $\mathscr{O}$ cut away by the sideline q_3, its isogonal conjugate the interior of the angle $\angle KA_3L$, the interior of the segment of $\mathscr{O}$ cut away by the sideline q_2, and the isogonal conjugate of this segment, the interior of the angle $\angle MA_2N$.

As a consequence, since the curve $\mathscr{K}_1$ is smooth, it must be tangent to the sideline q_2 at A_3, to the circumcircle $\mathscr{O}$ at A_1, and to the sideline q_3 at A_2.

By now we know how the sign of the function f_1 behaves. But we need to know the sign of the first affine coordinate of the pedal circumcenter $Q(E)$,

$$z_1 \,=\, z_1(E) \,=\, -\frac{f_1(E)}{2\mathcal{P}_{\mathscr{O}}(E)}\ ,$$

since it is this sign which tells us on which side of the sideline q_1 lies $Q(E)$: on the same side as the vertex A_1 if $z_1(E) > 0$, and on the opposite side if $z_1(E) < 0$. In Figure 12 the region where $z_1(E) > 0$ is gray, and the region where $z_1(E) < 0$ is white. In particular,

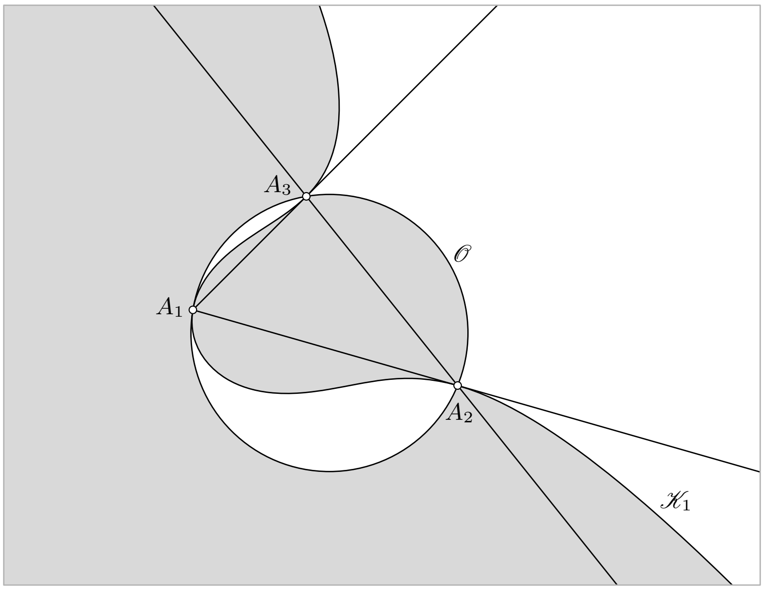

Figure 12. The sign of the first affine coordinate of the pedal circumcenter $Q(E)$: white is negative, gray positive.

we see that the segment cut from the circumcircle $\mathscr{O}$ by the sideline q_1 (on the side opposite to the vertex A_1) is gray, thus for a point E in the interior of this segment the corresponding pedal circumcenter lies on the same side of q_1 as A_1, that is, it lies on the side of q_1 opposite to E, whence it is the branch $\mathscr{H}_E$ of the inhyperbola $\mathscr{H}$, on the side of its focus E, which touches the sideline q_1, and it touches it at a point inside the line segment $\overline{A_2A_3}$. Now is the time to look again at Figure 8; notice, among other things, that the center Q of the hyperbola lies in the interior of the triangle Δ but outside its medial triangle.

Now suppose that the point E lies in the interior of the segment A_1SA_2 of the circumcircle $\mathscr{O}$ (Figure 13) and hence its isogonal conjugate F lies in the interior of the angle $\angle KA_3L$. The points E and F are the foci of an inhypebola $\mathscr{H}$ of the triangle Δ, with the branch $\mathscr{H}_E$ on the side of the focus E contained in the truncated angle MA_1A_2N and the branch $\mathscr{H}_F$ on the side of the focus F contained in the angle $\angle KA_3L$. The curve $\mathscr{K}_1$ cuts the interor of the circular segment A_1SA_2 into two subregions $\mathscr{U}_E$ and $\mathscr{V}_E$, and also cuts the interior of the angle $\angle KA_3L$ into two subregions $\mathscr{U}_F$ and $\mathscr{V}_F$, where the subregions $\mathscr{U}_E$ and $\mathscr{U}_F$ are coupled by isogonal conjugation, as are the subregions $\mathscr{V}_E$ and $\mathscr{V}_F$.

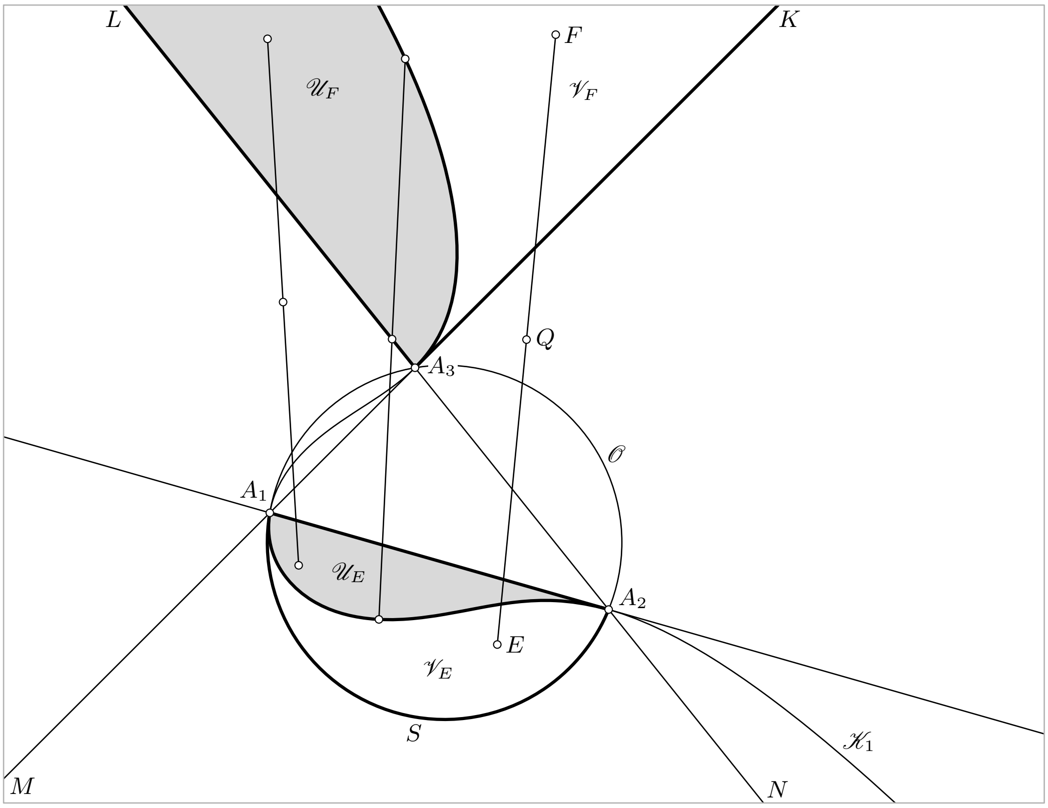

Figure 13

First suppose that the focus E of $\mathscr{H}$ lies in the subregion $\mathscr{V}_E$, and hence the focus F lies in the subregion $\mathscr{V}_F$. In this case the center Q of $\mathscr{H}$ lies on the side of the sideline q_1 opposite to the vertex A_1, that is, it lies on the side of q_1 opposite to the focus E; it follows that the hyperbola's branch $\mathscr{H}_E$ touches the sideline q_1 somewhere inside the half-line $\overrightarrow{A_2N}$.

Next, suppose that E lies in the subregion $\mathscr{U}_E$ and hence F lies in the subregion $\mathscr{U}_F$. Now the center Q of $\mathscr{H}$ lies on the same side of q_1 as A_1, so that it lies on the opposite side of q_1 to the focus F, and it is the hyperbola's branch $\mathscr{H}_F$ which touches q_1 inside the half-line $\overrightarrow{A_3L}$.

Finally, if the focus E is inside the arc A_1A_2 of the curve $\mathscr{K}_1$, then the focus F is on the part of the curve $\mathscr{K}_1$ in the interior of the angle $\angle KA_3L$, and the center Q of $\mathscr{H}$ lies on the sideline q_1 which is an asymptote of the inhyperbola $\mathscr{H}$, that is, $\mathscr{H}$ touches q_1 in its point at infinity.

Once again we are going back to take another look at the situation in Figure 8, only this time we are also drawing the curves $\mathscr{K}_2$ and $\mathscr{K}_3$, getting Figure 14. See where the focus E of the hyperbole $\mathscr{H}$ lies relative to $\mathscr{K}_2$ and $\mathscr{K}_3$: the positioning of E explains why the sidelines q_2 and q_3 are touching the branch $\mathscr{H}_F$ of the hyperbola.

It is high time we draw a figure with all three curves $\mathscr{K}_1$, $\mathscr{K}_2$, $\mathscr{K}_3$ in it: this is Figure 15. With (i,j,k) a permutation of $(1,2,3)$, the curves $\mathscr{K}_i$ and $\mathscr{K}_j$ intersect in two isogonally conjugate points G_k and H_k, which lie on the internal angle bisector at the vertex A_k of the triangle Δ on the opposite sides of the vertex, both at the

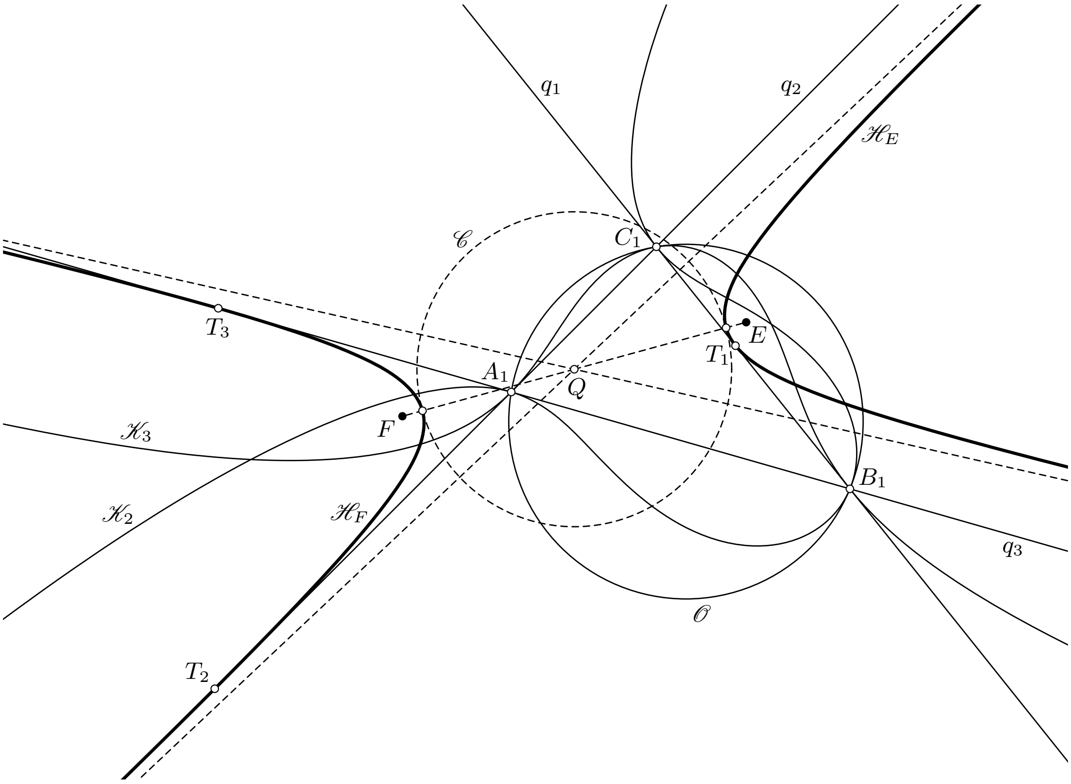

Figure 14

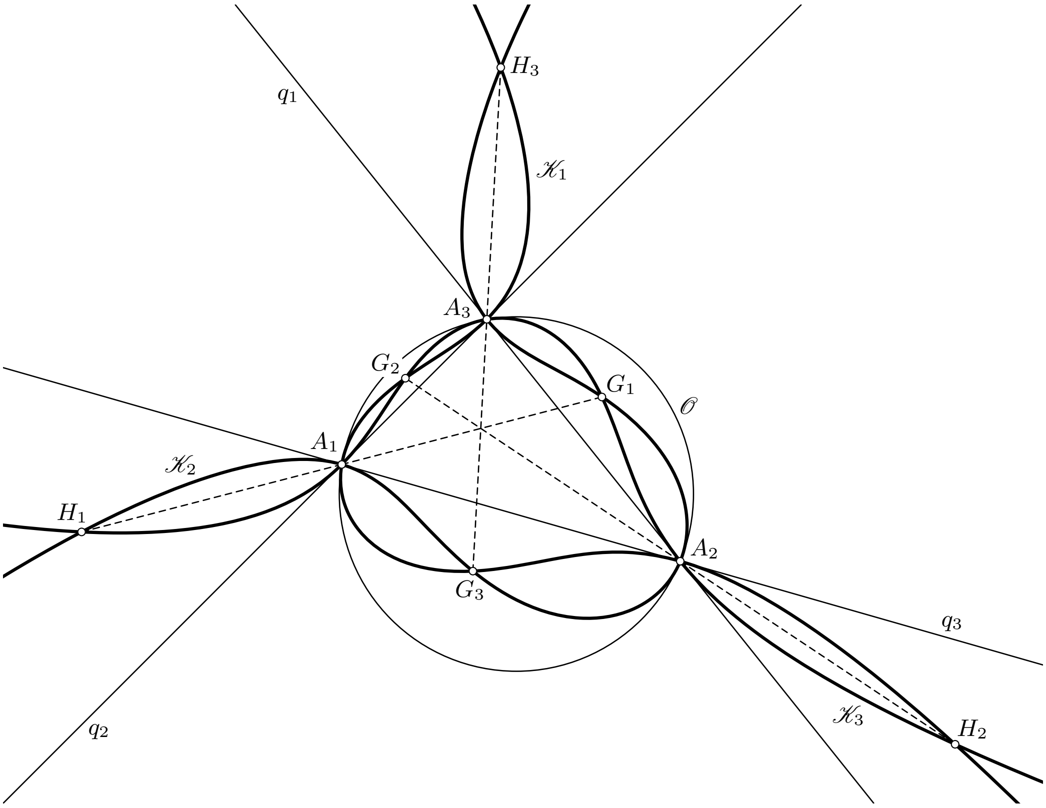

Figure 15

distance $\sqrt{|A_kA_i|\cdot|A_kA_j|}$ from it. If $E = G_k$, then $F = H_k$ and $Q = A_k$, the sidelines q_i and q_j are the asymptotes of the inhyperbola $\mathscr{H}$, and the branch $\mathscr{H}_E$ of the inhyperbola touches the sideline q_k inside the line segment $\overline{A_iA_j}$. Figure 16 has the inhyperbola $\mathscr{H}$ for $E = G_3$ and $F = H_3$.

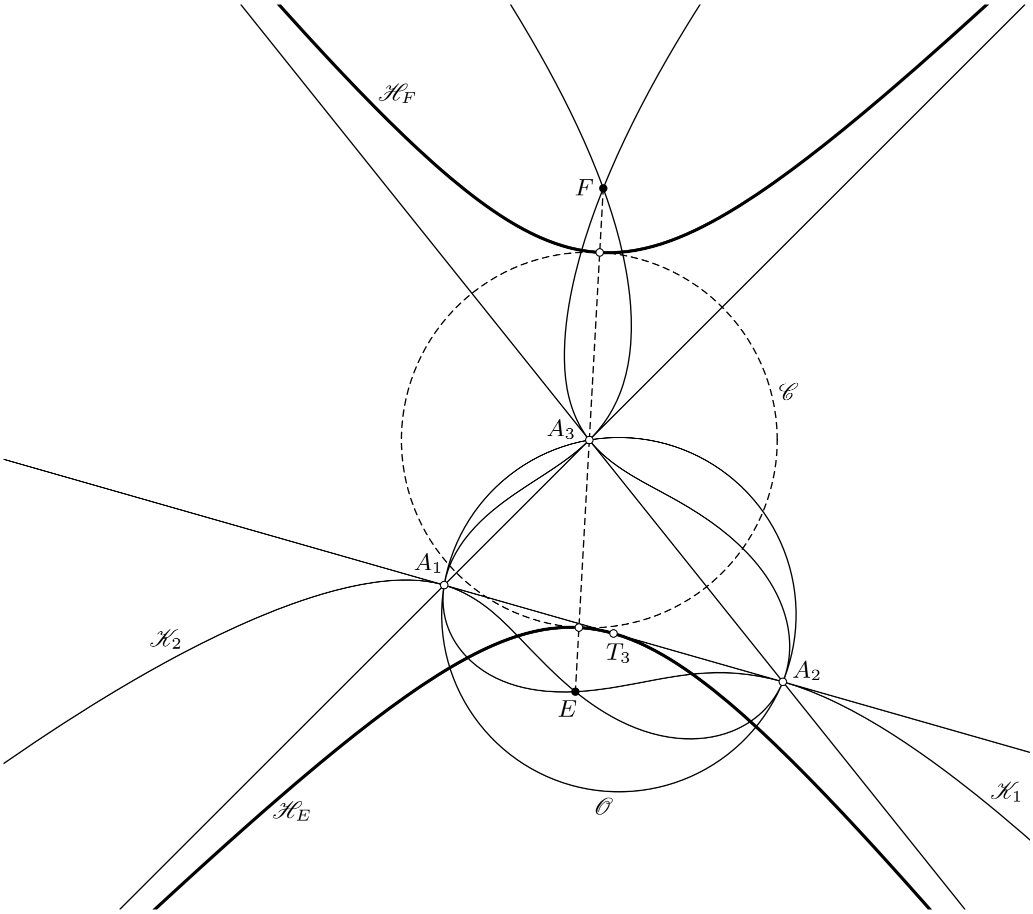

Figure 16

In footnote 3 on page 4 we promised to discuss later the special case, which we avoided up to now, where the point E lies on the circumcircle $\mathscr{O}$ of the triangle Δ, but is different from the vertices of the triangle; that "later" has arrived. In this special case the pedal points of E are collinear, and the line through them — the Simson line — is the degenerate pedal circumcircle $\mathscr{C}(E)$ of the point E. Since the center of the line $\mathscr{C}(E)$, where this line is regarded as a circle, lies on the line at infinity in the direction orthogonal to the direction of the line, the conic is a parabola $\mathscr{P}$, which is constructed as in Figure 17: $\mathscr{C} = \mathscr{C}(E)$ is the Simson line, the directrix d of the parabola $\mathscr{P}$ is parallel to $\mathscr{C}$, the axis q of the parabola is perpendicular to $\mathscr{C}$, and V, the intersection point of q and $\mathscr{C}$, is the vertex of the parabola. It is evident, from Figure 17, how the points of tangency of the inparabola $\mathscr{P}$ on the sidelines of the triangle Δ are constructed: it is the same construction as for an inellipse or an inhyperbola, except that one focus, the isogonal conjugate of the focus E, is a point at infinity.

It remains to discuss the situation, which we also shunned so far, where the point E lies on one of the sidelines of the triangle Δ. If E lies, say, on the sideline q_3, but is

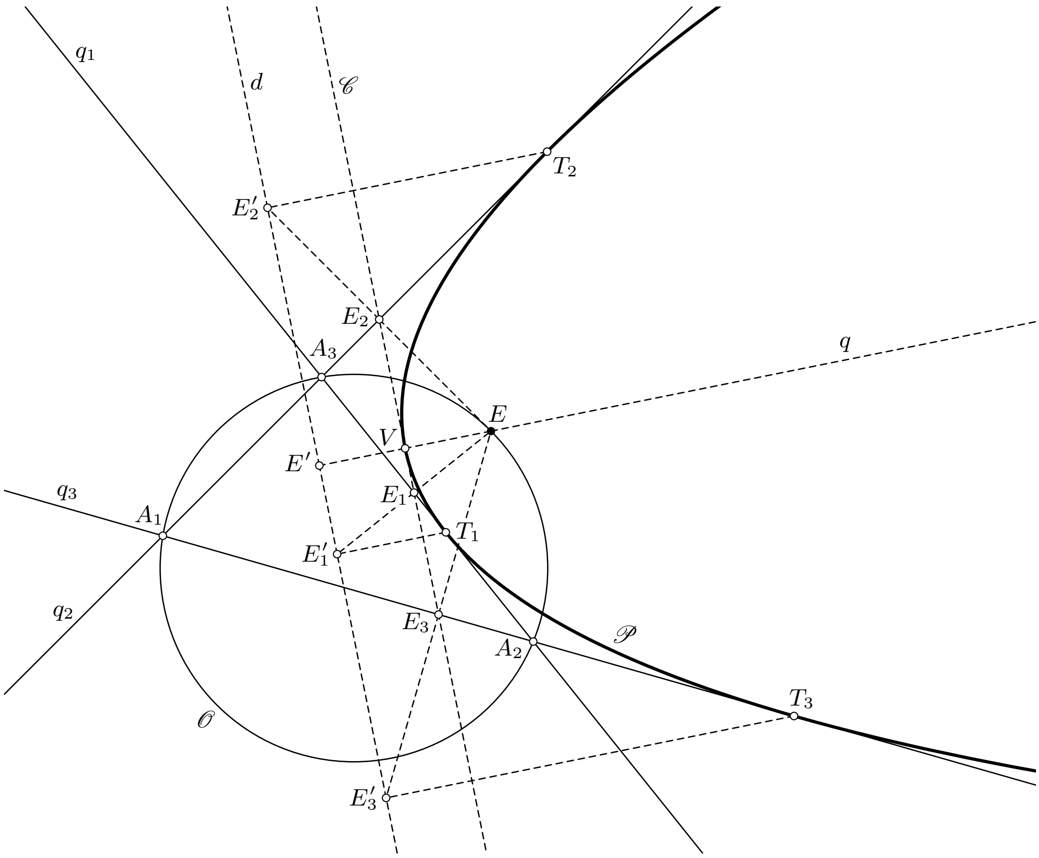

Figure 17

different from the vertices A_1 and A_2, then the E is its own pedal point E_3 and its pedal circumcenter is the midpoint of the line segment $\overline{EA_3}$ (Figure 18). In this case the inconic

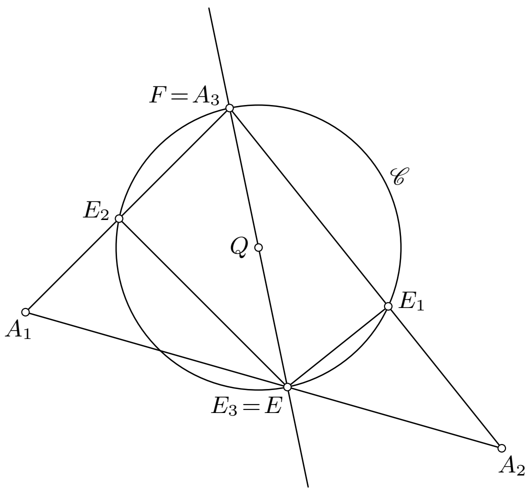

Figure 18

associated with the point E is degenerated, it is part of the line $\overleftrightarrow{EA_3}$. We determine the associated inconic by a limiting process where the point E approaches its final position on the sideline from outside the sideline. The result depends on the side of the sideline q_3

from which the point E approaches it. If E approaches q_3 from its side containing the vertex A_3, then the limit inconic is the line segment $\mathscr{E} = \overline{EA_3}$, a degenerate ellipse (Figure 19). Otherwise, if E approaches q_3 from the side opposite the one containing

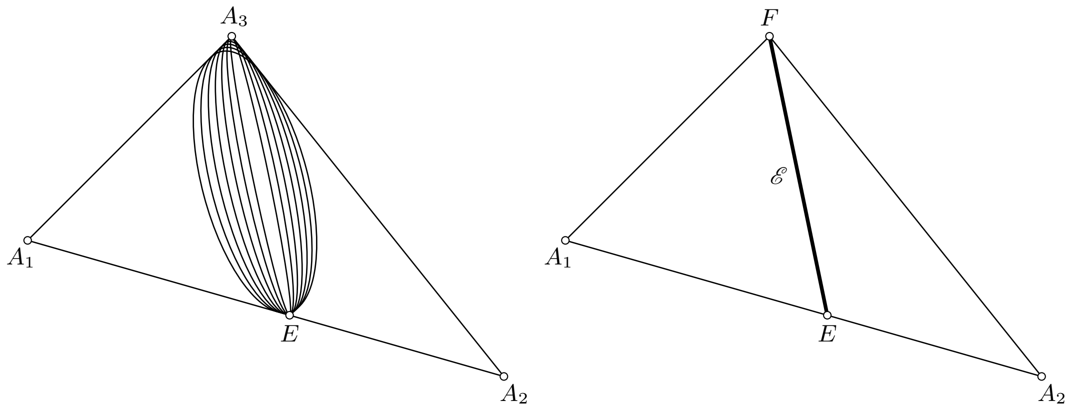

Figure 19

the vertex A_3, then the limit inconic consists of two half-lines pointing away from the line segment $\overline{EA_3}$, which are the two branches $\mathscr{H}_E$ and $\mathscr{H}_F$ of the degenerate limit inhyperbola $\mathscr{H}$ (Figure 20).[6]

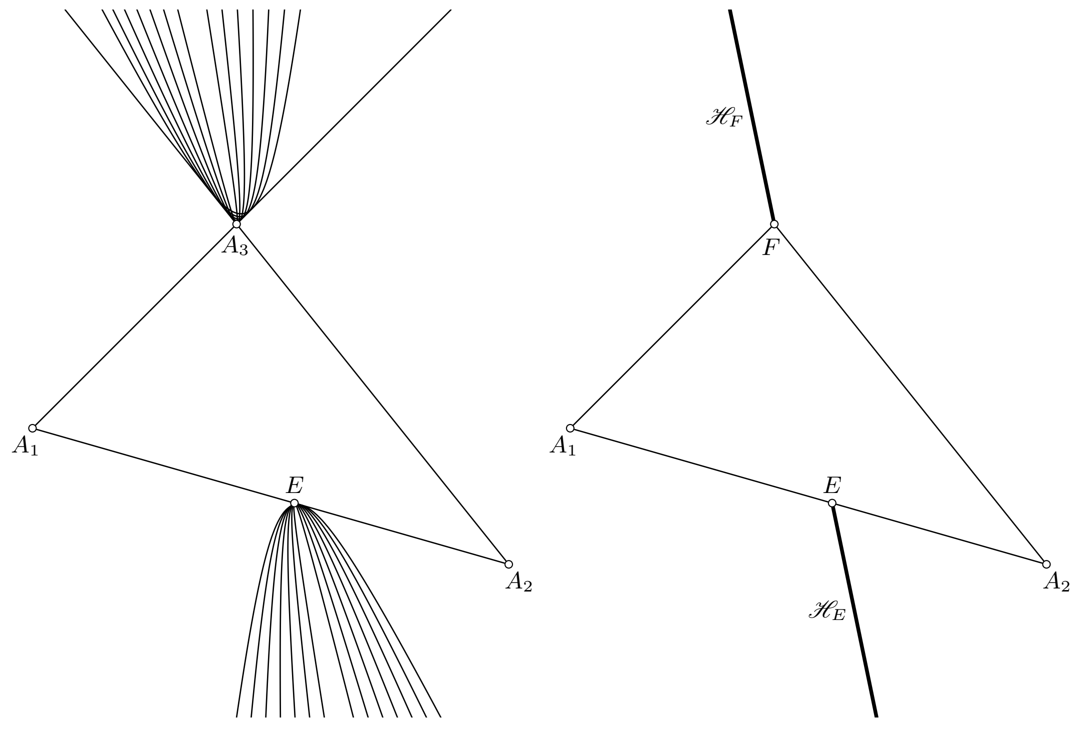

Figure 20

[6]Let $\mathscr{F}$ be a filter on a set X, and for each $x \in X$ let $\mathscr{A}_x$ be a subset of the plane. We say that the sets $\mathscr{A}_x$ converge on the filter $\mathscr{F}$ to a subset $\mathscr{B}$ of the plane if for every bounded subset $\mathscr{Q}$ of the plane and for every $\delta > 0$ there exists a set $U \in \mathscr{F}$ such that for every $x \in U$ the δ-neighborhood of $\mathscr{B}$ contains $\mathscr{A}_x \cap \mathscr{Q}$ and the δ-neighborhood of $\mathscr{A}_x$ contains $\mathscr{B} \cap \mathscr{Q}$. If the sets $\mathscr{A}_x$ converge to some closed set $\mathscr{B}$, then $\mathscr{B}$ is unique.

There are also degenerate limit inparabolas. If the point E approaches the vertex A_3 along the arc $\overset{\frown}{A_2A_3}$ of the circumcricle $\mathscr{O}$, then the associated parabola converges to the half-line $\mathscr{P}$ parallel to the sideline q_3, starting at the vertex A_3, and pointing in the direction from the vertex A_1 to the vertex A_2 (Figure 21).

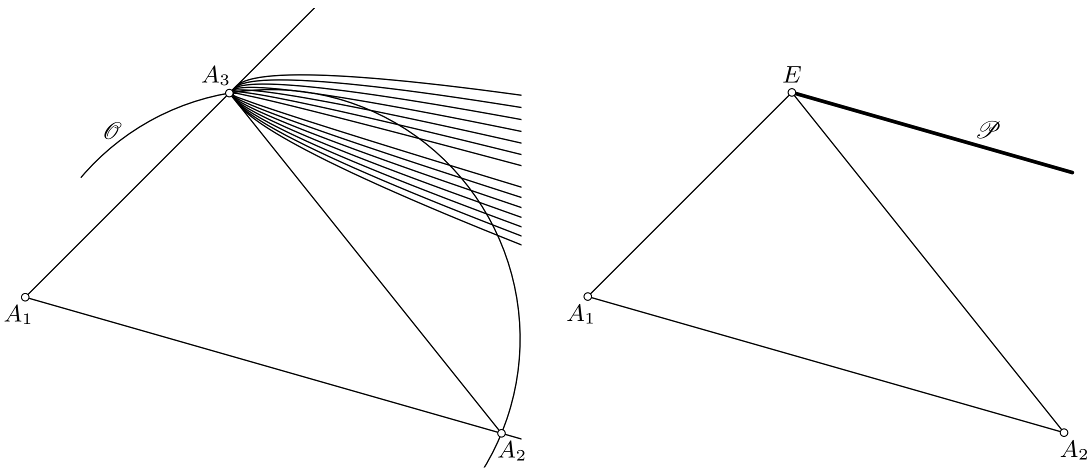

Figure 21

We have now a complete picture of what kind of inconic of the triangle Δ is associated with its focus E, depending on where in the plane the point E lies; see Figure 22, where the central white triangle is the triangle Δ.

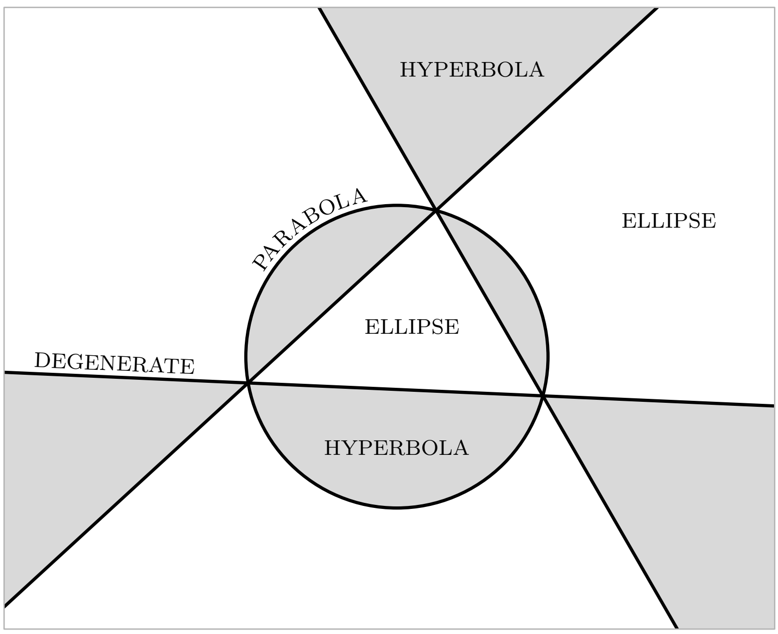

Figure 22

We conclude with an observation whose character is rather that of a mild curiosity. If the point E is a vertex of the triangle Δ, say the vertex A_3, then its degenerate pedal triangle has $E_1 = E_2 = E$, while the third pedal point E_3 is the foot of the altitude dropped from the vertex A_3 to the sideline q_3. In this case each circle $\mathscr{C}$ with the chord $\overline{EE_3}$ (which may be a degenerate circle, the line $\overleftrightarrow{EE_3}$) is a circumcircle of the pedal triangle (Figure 23). The point $E = A_3$ in and of itself therefore does not have

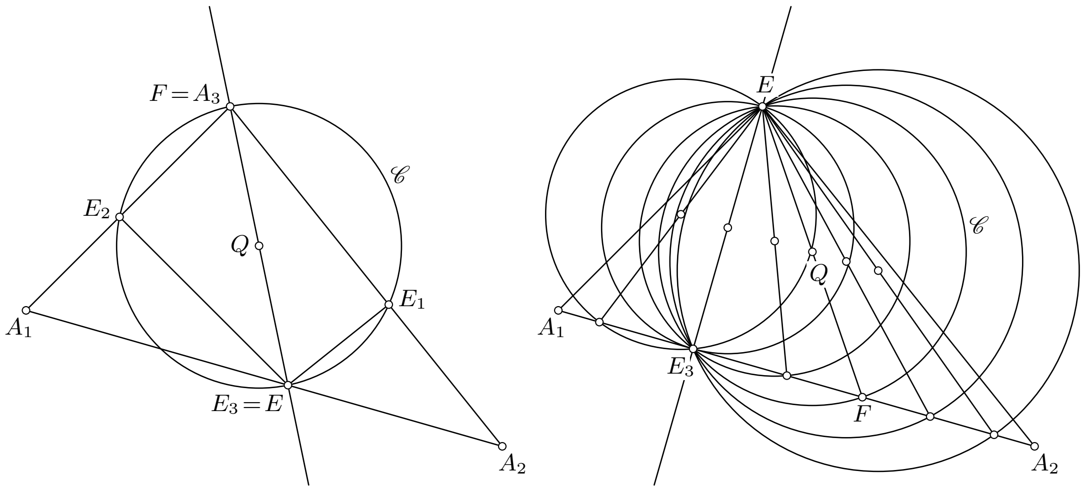

Figure 23

a unique isogonal conjugate, since every point F on the sideline q_3 can be considered its isogonal conjugate. However, suppose that the point E approaches the vertex A_3 along some fixed line l laid through this vertex. Then the isogonal conjugate F of E moves along the line l' obtained by reflecting the line l on the angle bisectors p and q of the triangle Δ at its vertex A_3, and approaches the point of intersection of the line l' with the sideline q_3 (which is the point at infinity on q_3 in case the line l is tangent to the circumcircle $\mathscr{O}$). See Figure 24. If the line l is given as the line connecting some

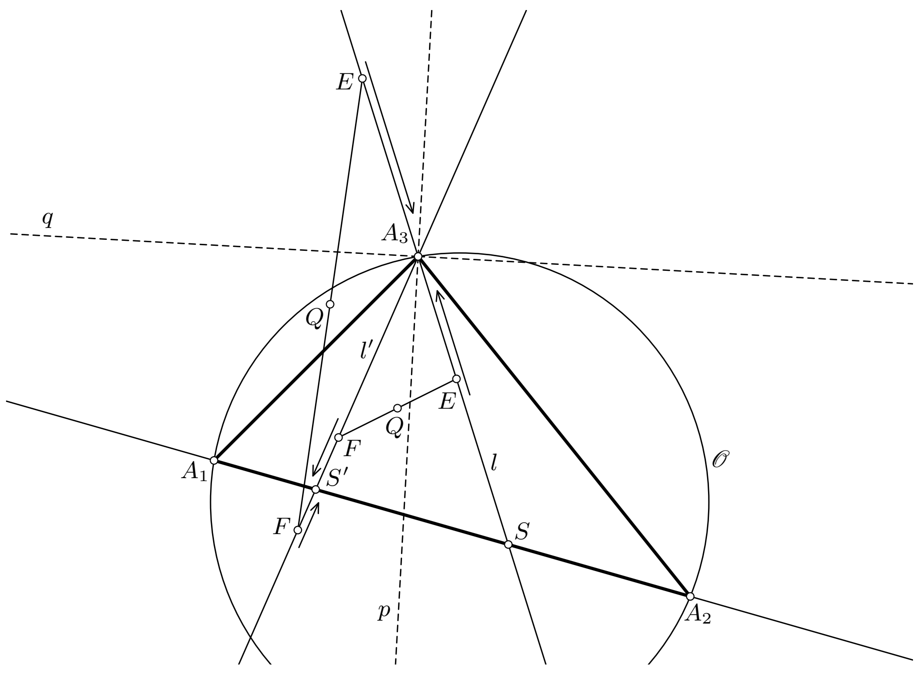

Figure 24

point S on q_3 to the vertex A_3, and S' is the intersection point of the lines l' and q_3, then the transformation $S \mapsto S'$ is a projective involution of the line q_3 (augmented with its point at infinity). There is a fun construction of the point S' from a given point S, presented in Figure 25: we construct first the point X on the perpendicular

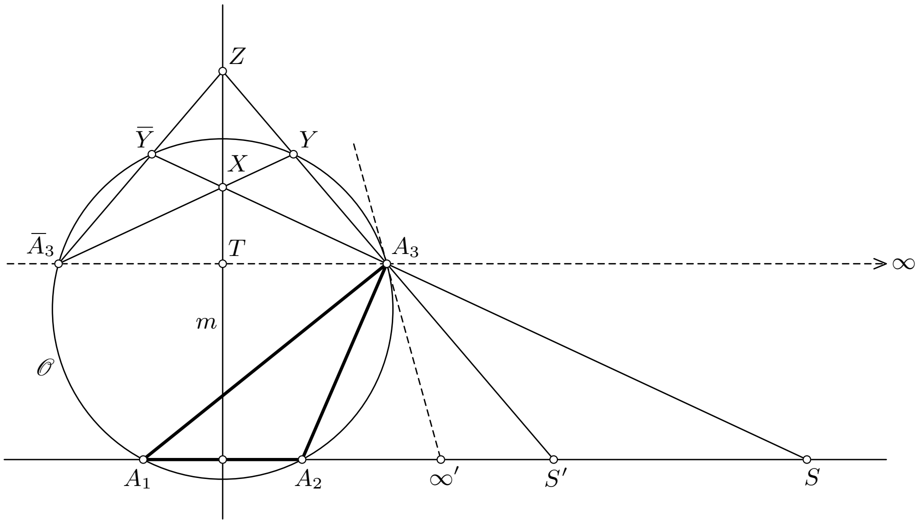

Figure 25

bisector m of the side $\overline{A_1A_2}$, next the point Y on the circumcircle $\mathscr{O}$ of the triangle, and then the point S' on the sideline q_3. Starting with the point S' the construction produces, in order, the points Z, $\overline{Y}$, and S: we get back to the point S because $S \mapsto S'$ is an involution. In particular, if we start with the point ∞ at infinity on the sideline q_3, the construction gives first the point T then the point A_3; now the line through the duplicate point A_3 on the circumcircle $\mathscr{O}$ is of course the tangent of the circumcircle at A_3, which intersects the sideline q_3 at the point ∞'. (How the construction proceeds if we start with the point ∞'?)

References

[1] J. Steiner, "Développement d'une série de théorèmes relatifs aux sections coniques." *Annales de Mathématiques*, Vol. 19, pp. 37–64; Jacob Steiner's Gesammelte Werke, Vol 1, pp. 191–210.