Quadratrix of Hippias and Viète's infinite product

Contemplating the arc of the quadratrix of Hippias, from the point $\mathrm{C} = (0, c)$ to the

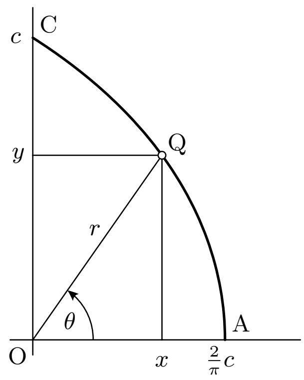

Figure 1: The quadratrix of Hippias.

point $\mathrm{A} = \left(\frac{2}{\pi}c, 0\right)$ (Figure 1), the factor $\frac{2}{\pi}$ reminded me of the Viète's infinite product

$$\frac{2}{\pi} = \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{2+\sqrt{2}}}{2} \cdot \frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2} \cdot \frac{\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}}{2} \cdots .$$

As you shall see, this was not just an idle reminiscence.

According to the construction of the quadratrix, $y = c \cdot \left(\theta / \frac{\pi}{2}\right) = 2c\theta/\pi$. Since $r = y/\sin\theta$, we have the polar equation of the point Q on the quadratrix:

$$r = \frac{2c}{\pi} \frac{\theta}{\sin\theta} . \tag{1}$$

We get the cartesian equation by substituting $\theta = \pi y/2c$ into $x = r(\theta)\cos\theta$:

$$x = y \cdot \cot\left(\frac{\pi}{2c} y\right) .$$

Note that the construction is indeterminate at $\theta = 0$; the point A is determined by continuity, as the limit point of $\mathrm{Q}(\theta)$ as $\theta \to 0$. As a traditional alternative, we may

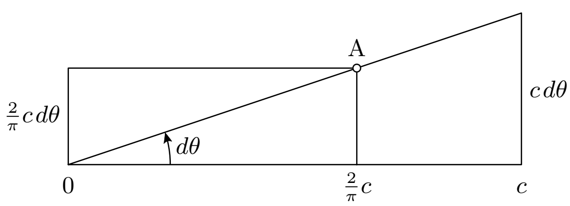

Figure 2: The point $\mathrm{A} = \mathrm{Q}(d\theta)$.

invoke infinitesimals and define the point A as $\mathrm{Q}(d\theta)$ (Figure 2).

When we try to construct a point on the quadratrix at a small angle θ, with a pencil on paper (or a stick in sand), we find that the intersection point — of the line through the origin in the direction θ and the line parallel to the x-axis at the ordinate $2c\theta/\pi$ — is ill-defined. Is there a more reliable construction of those points on the quadratrix that are close to the x-axis? There is such a construction, and it is related to the Viète's product.

The quadratrix of Hippias can be used to divide an acute angle into any prescribed number of equal parts; in particular, it can be used to trisect angles. It can be also used to *bisect* angles. This application of the quadratrix, which is a rather exotic curve, may appear slighly perverse, since we already know how to bisect an angle using only straightedge and compass; but let us do it, anyway. Looking at Figure 3 we see that,

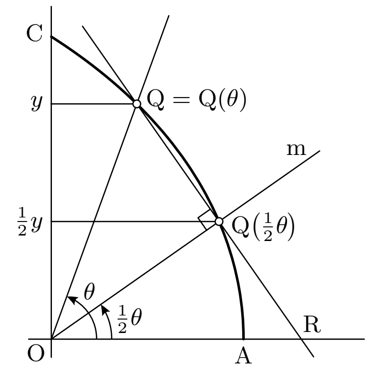

Figure 3: Bisecting an angle using the quadratrix of Hippias.

given a point $\mathrm{Q} = \mathrm{Q}(\theta)$ on the quadratrix, we obtain the point $\mathrm{Q}\big(\frac{1}{2}\theta\big)$ by intersecting the bisector m of the angle $\angle\mathrm{OAQ}$[1] by its perpendicular QR laid through the point Q (this is so because the triangle $\triangle\mathrm{RQO}$ is isosceles). We can therefore turn the tables, and instead of using the quadratix to bisect the angle $\angle\mathrm{OAQ}$, we use the bisection of this angle to determine the point $\mathrm{Q}\big(\frac{1}{2}\theta\big)$ on the quadratrix from a known point $\mathrm{Q}(\theta)$ on it. When this construction of the point $\mathrm{Q}\big(\frac{1}{2}\theta\big)$, for a small angle θ, is carried out by a pencil, it is much more precise than the defining construction as the intersection point of the bisector m and the line parallel to the x-axis at half the ordinate of the point $\mathrm{Q}(\theta)$.

Let P be any point in the plane not on the non-negative half of the x-axis. The point P has unique polar coordinates (r, θ) with $r > 0$ and $-\pi < \theta < \pi$. Draw the line through the origin O in the direction $\frac{1}{2}\theta$, and drop a perpendicular to it from the point P

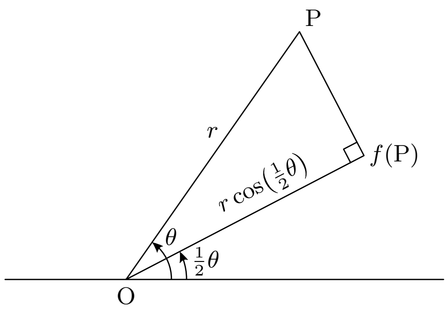

Figure 4: The angle-halving transformation f.

to the point $f(\mathrm{P})$ (Figure 4); polar coordinates of the point $f(\mathrm{P})$ are $\big(r\cos\big(\frac{1}{2}\theta\big), \frac{1}{2}\theta\big)$.

[1] I write $\angle\mathrm{OXY}$ to denote the (oriented) angle from OX to OY.

Starting with a point P, we repeatedly apply the transformation f, producing a sequence of points P, $\mathrm{P}_1 = f(\mathrm{P})$, $\mathrm{P}_2 = f\big(f(\mathrm{P})\big) = f^2(\mathrm{P})$, ..., $\mathrm{P}_n = f^n(\mathrm{P})$, ..., which converges

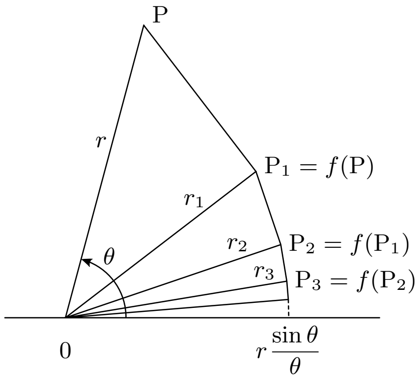

Figure 5: The sequence of iterated f-transforms of a point P.

to the point on the x-axis with the abscissa $r \cdot (\sin\theta)/\theta$ (Figure 5), because

$$\cos\frac{\theta}{2}\cdot\cos\frac{\theta}{4}\cdots\cos\frac{\theta}{2^n}\cdots \;=\; \frac{\sin\theta}{\theta}\;. \tag{2}$$

This product formula (one of Euler's) is easy to prove; we just repeatedly apply the formula $\sin\alpha = 2\sin(\alpha/2)\cos(\alpha/2)$:

$$\begin{aligned}
\frac{\sin\theta}{\theta} &= \cos\frac{\theta}{2}\cdot\frac{\sin(\theta/2)}{\theta/2}\\
&= \cos\frac{\theta}{2}\cdot\cos\frac{\theta}{4}\cdot\frac{\sin(\theta/4)}{\theta/4}\\
&\cdots\quad\cdots\quad\cdots\\
&= \cos\frac{\theta}{2}\cdot\cos\frac{\theta}{4}\cdots\cos\frac{\theta}{2^n}\cdot\frac{\sin(\theta/2^n)}{\theta/2^n}\;.
\end{aligned}$$

With n increasing to infinity, $\theta_n = \theta/2^n$ converges to 0 and $(\sin\theta_n)/\theta_n$ converges to 1 (this is true for any nonzero *complex* number θ). Polar coordinates of the point P_n are $(r_n,\, \theta/2^n)$, where

$$r_n \;=\; r\cdot\cos\frac{\theta}{2}\cdot\cos\frac{\theta}{4}\cdots\cos\frac{\theta}{2^n} \;=\; r\cdot\frac{\sin\theta}{\theta}\cdot\frac{\theta/2^n}{\sin(\theta/2^n)}\;,$$

thus all points P_n lie on the quadratrix with the polar equation (1), where the parameter c is $\frac{\pi}{2}r\cdot(\sin\theta)/\theta$.

To compute the factors $\cos(\theta/2^n)$ in the Euler's product (2), we use the formula $\cos(\alpha/2) = \sqrt{\frac{1}{2}(1+\cos\alpha)}$, which we rewrite as $2\cos(\alpha/2) = \sqrt{2+2\cos\alpha}$, to get the following recurrence relations for $c_n = 2\cos(\theta/2^n)$:

$$c_0 \;=\; 2\cos\theta\;, \qquad c_{n+1} \;=\; \sqrt{2+c_n}\;.$$

The Euler's product formula is then

$$\frac{\sin\theta}{\theta} \;=\; \frac{c_1}{2}\cdot\frac{c_2}{2}\cdots\frac{c_n}{2}\cdots\;.$$

Taking $\theta = \frac{1}{2}\pi$, we get the Viète's infinite product. With $\theta = \frac{1}{6}\pi$ we obtain an infinite product formula for $3/\pi$ that is a worthy companion to the Viète's product:

$$\frac{3}{\pi} = \frac{\sqrt{2+\sqrt{3}}}{2} \cdot \frac{\sqrt{2+\sqrt{2+\sqrt{3}}}}{2} \cdot \frac{\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{3}}}}}{2} \cdots .$$

The partial products $r_0 = 1$, r_1, r_2, ... of the Viète's product are the distances from the origin of the points C, $f(\mathrm{C})$, $f^2(\mathrm{C})$, ... (Figure 6), which march, in ever smaller steps,

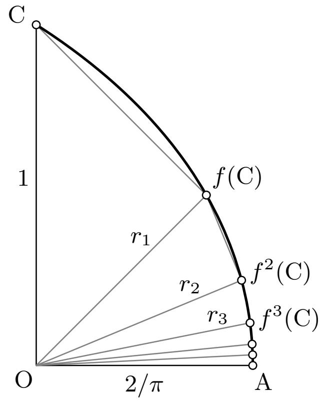

Figure 6: The Viète's infinite product visualized.

downwards along the quadratrix (with the parameter $c = 1$), approaching the point A. How fast do r_n converge to $2/\pi$? Since

$$r_n = \frac{2}{\pi}\frac{\frac{\pi}{2}/2^n}{\sin\left(\frac{\pi}{2}/2^n\right)} = \frac{2}{\pi}\left(1 + \frac{\pi^2}{24}2^{-2n} + \mathrm{O}(2^{-4n})\right) ,$$

each iteration divides the relative error by four, approximately, which means that we gain roughly six decimal places of precision with every ten iterations. Numerical checks are in agreement with this estimate: $\frac{\pi}{2}r_5 = 1.000\,401\ldots$, $\frac{\pi}{2}r_{15} = 1.000\,000\,000\,382\ldots$, $\frac{\pi}{2}r_{25} = 1.000\,000\,000\,000\,000\,365\ldots$, $\frac{\pi}{2}r_{35} = 1.000\,000\,000\,000\,000\,000\,000\,348\ldots$.

We pause here to look back at Euler's product formula (2). Suppose we are faced with the task of proving that it holds for every complex number θ, but we do not hit at once on the idea of using the formula $\sin\alpha = 2\sin(\alpha/2)\cos(\alpha/2)$. How can we go about proving the formula (2)? One way to do it is to gather pieces of information about the function $\sigma(z)$ of complex variable defined as the infinite product

$$\sigma(z) = \prod_{n=1}^{\infty} \cos\frac{z}{2^n} , \qquad z \in \mathbb{C} ,$$

until we know enough about it to prove Euler's product formula. First, $\sigma(z)$ is an entire function because the product converges uniformly on compact subsets of the complex plane: if K is a compact subset of $\mathbb{C}$, then there exists a positive constant M_K so that $\left|\cos(z/2^n)-1\right| \leqslant M_K/2^{2n}$ for every $z \in K$. Second, $\sigma(z)$ is an even function of z. Third, $\sigma(z)$ satisfies the conditions

$$\sigma(0) = 1 , \qquad \sigma(2z) = \sigma(z)\cos z \quad \text{for every } z \in \mathbb{C} ,$$

and it is the only even entire function that satisfies these conditions. To prove the latter, we substitute $\sigma(z) = \sum_{n=0}^{\infty} s_n z^{2n}$ into the conditions, expand the right hand side of the functional equation, and by equating coefficients at z^{2n} obtain the recurrence relations

$$s_0 = 1 , \qquad (2^{2n} - 1) \cdot s_n = \text{linear combination of } s_{n-1}, \ldots, s_1, s_0 \quad \text{for } n > 0 ,$$

where the coefficients of linear combinations are known rational numbers. And that's it: we have all we need. We simply verify that the even entire function $\sigma(z) = (\sin z)/z$ satisfies the conditions: $\sigma(0) = \lim_{z\to 0}(\sin z)/z = 1$, so certainly $\sigma(2{\cdot}0) = 1 = \sigma(0)\cos 0$, and $\sigma(2z) = \sigma(z)\cos z$ for a nonzero z because $\sin 2z = 2\sin z\cos z$. Done.