The approximate regular nonagon in Wikipedia

France Dacar, Jožef Stefan Institute
France.Dacar@ijs.si

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Wikipedia's article "Nonagon" gives the following approximate construction of a regular nonagon[1] (Figure 1): D is the midpoint of OB, |BE| = |BD|, F and G are the

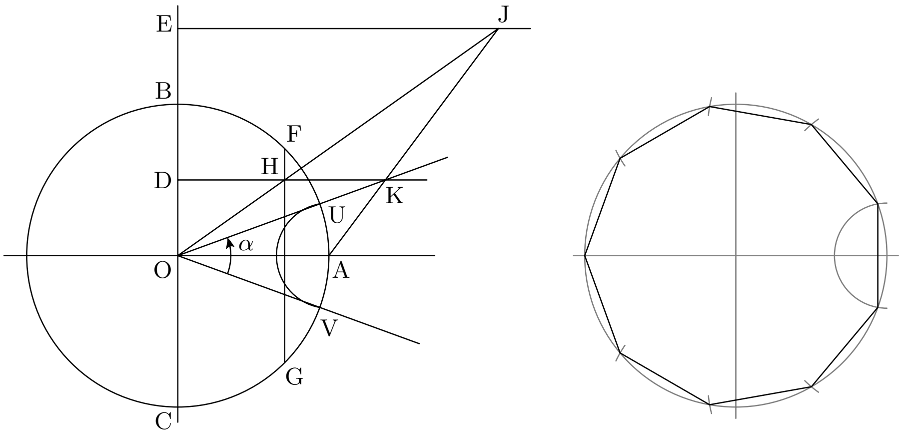
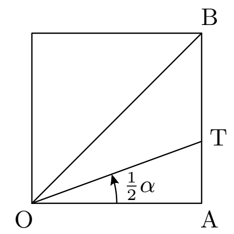


Figure 1: The approximate construction of a regular nonagon in Wikipedia.

midpoints of the arcs AB resp. AC, and it is clear from the figure how the points H, J, K, U, and V, in this order, are constructed. The angle $\alpha = \measuredangle \mathrm{OVU} \doteq 39.999\,068^\circ$ is very close to the desired 40° angle. The indicated construction of nonagon's vertices ticks off seven sides whose central angles are α, while the remaining two sides have central angles $(360^\circ - 7\alpha)/2 \doteq 40.003\,260^\circ$. If we tick off α nine times, the remaining angle is $360^\circ - 9\alpha \doteq 0.008\,382^\circ \doteq 30.17''$; as long as $|\mathrm{OA}| \leqslant 683.5\,\mathrm{mm}$, the remaining gap is less that 0.1 mm, which is certainly good enough for drawing purposes. Notice that we can halve even this small error by using the angle $\beta = 60^\circ - \frac{1}{2}\alpha \doteq 40.000\,466^\circ$ instead of $\alpha = 2 \cdot \frac{1}{2}\alpha$. Since β is larger then 40°, nine angles β overshoot 360° by $9\beta - 360^\circ \doteq 0.004\,191^\circ \doteq 15.09''$, which gives an overlap less than 0.1 mm for $|\mathrm{OA}| \leqslant 1367\,\mathrm{mm}$.

[1]The construction is presented by an attractive animation.

It is clear from the construction that the points A, B, ..., J, K have coordinates in $\mathbb{Q}(\sqrt{2})$, thus also $\tan\left(\frac{1}{2}\alpha\right) = y_\mathrm{K}/x_\mathrm{K} \in \mathbb{Q}(\sqrt{2})$. And indeed, $\tan\left(\frac{1}{2}\alpha\right) = 9/\sqrt{2} - 6$,



$$|\mathrm{AT}| = \frac{9}{2}\,|\mathrm{OB}| - 6\,|\mathrm{OA}|$$

Figure 2: The approximate construction from Wikipedia, simplified.

which suggests the simpler construction of the angle $\frac{1}{2}\alpha$ shown in Figure 2. This simplified construction is somewhat cumbersome, because we must carry out the geometric computation of $|\mathrm{AT}|$ with care, lest we march off the piece of paper on which we are doing the construction. Moreover, the segment AT is short in comparison with the line segments we have to add and subtract to obtain it, so the relative drawing error (caused by imprecise use of compasses etc.) is uncomfortably large. To remedy this, we rewrite $9/\sqrt{2} - 6$ as the fraction $3/\left(4+3\sqrt{2}\right)$, which is realized by the construction

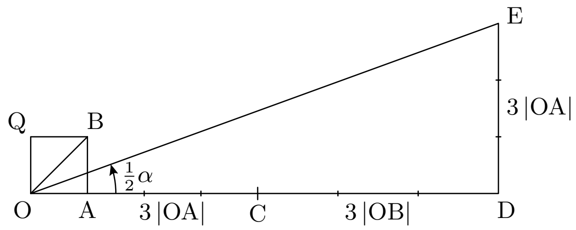


Figure 3: Another simplification ot the approximate construction from Wikipedia.

in Figure 3. Now the overall relative drawing error is the same as the relative error of a single transfer of a segment length. We are still not satisfied with the construction, since it rather spreads away from the starting square OABQ. Rewriting, once more, $3/\left(4+3\sqrt{2}\right)$ as $\frac{3}{4}/\left(1+\frac{3}{4}\sqrt{2}\right)$, we in effect scale up the square by factor four, as shown in

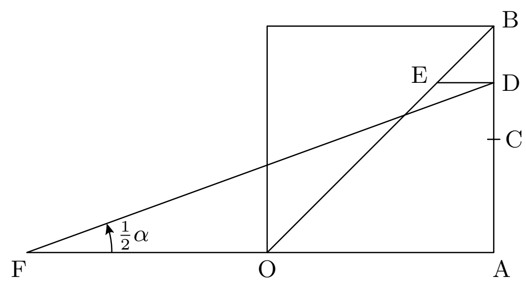


Figure 4: Yet another simplification ot the approximate construction from Wikipedia.

Figure 4: C is the midpoint of AB, D is the midpoint of CB, ED is parallel to OA, and $|\mathrm{OF}| = |\mathrm{OE}|$. Figure 4 does not show the whole construction; one also has to construct the square and the two midpoints, but these auxiliary constructions are omitted, since they would only clutter up the figure. However, suppose that we have the points O, A, D, and E already pre-constructed, say on graph paper as in Figure 5 (where the point E

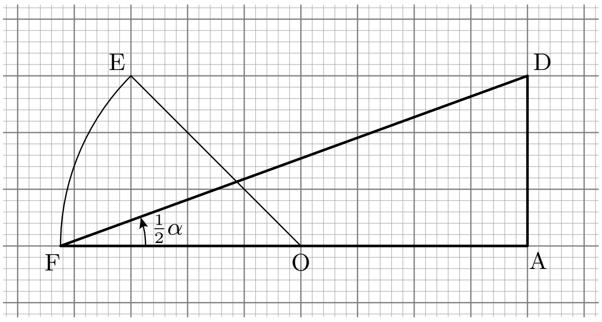


Figure 5: A construction of the angle $\frac{1}{2}\alpha$ on graph paper.

has been moved to a more convenient position); then all it remains to construct is the point F. Now, it's a safe bet that *this* construction cannot be simplified any further.

What thought processes led to the construction in Wikipedia? Of this I have not the slightest idea. It is highly improbable that someone hit upon $9/\sqrt{2}-6$ as a surprisingly good approximation of $\tan(20°)$ and then devised the attractive construction of the angle $\frac{1}{2}\alpha = \arctan(9/\sqrt{2}-6)$ in the article "Nonagon", because surely one would prefer to give some simpler construction suggested by the number $9/\sqrt{2}-6 = 3/(4+3\sqrt{2})$, possibly even the graph-paper construction of Figure 5 — which is what I would do.

It is fruitless to speculate about the origins of the construction in Wikipedia. What we can do is this: having seen one approximate graph-paper construction of the 20° angle, we may search for more constructions of this sort. We need not restrict ourselves to that one angle, we may seek good approximations of any of the angles 40°, 20°, 10°, and 5°; in each case we can then construct the angle $40° = 60°-20° = 30°+10° = 45°-5°$ to the same accuracy as the angle we have actually aproximated.

First we shall look for approximations t of $\tan(\varphi)$, $\varphi = 40°, 20°, 10°, 5°$, of the form

$$t = \frac{k + \sigma\sqrt{m^2+n^2}}{p + \tau\sqrt{q^2+r^2}} ,$$

where k and p are integers, factors σ and τ take values in $\{-1, 0, 1\}$, and m, n, q, r are positive integers such that m^2+n^2 and q^2+r^2 are not squares of integers; moreover, we require the denominator to be positive (and then also the numerator of a good enough approximation will be positive). We put an upper bound M on $|k|$, m, n, $|p|$, q, and r, and then search for the most accurate approximation, where the accuracy of an approximation t of $\tan(\varphi)$ is defined as $|\arctan(t)-\varphi|$. Choosing $M = 25$, the best approximation is found for the angle $\varphi = 40°$, and it is

$$t_{40} = \frac{-10+\sqrt{21^2+24^2}}{9+\sqrt{6^2+16^2}} = \frac{-10+3\sqrt{113}}{9+2\sqrt{73}} ,$$

with the error $\varepsilon_{40} := \arctan(t_{40}) - 40° \doteq -0.000\,118''$. Figure 6 shows the graph-paper construction of the angle $\beta = \arctan(t_{40})$.

The angle β is a rather accurate approximation of 40°. But, just how surprising is the accuracy of this approximation? The precision (i.e. the number of reliable decimal digits) of the approximation is $-\log_{10}\big(|\varepsilon_{40}|/40°\big) \doteq 9.09$. To achieve this precision we have

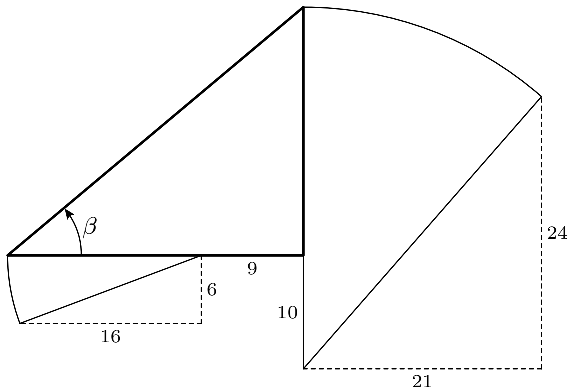


Figure 6: An approximate graph-paper construction of the 40° angle.

to know the numbers 10, 21, 24, 9, 6, and 16, and besides that, we also have to know how the angle β is constructed out of these numbers. All in all we have to know 10 digits (among other things) to construct the angle 40° to precision of 9 digits. According to this comparison the accuracy we have achieved is anything but surprising. However, counting the digits that are actually needed to write the six numbers is a very crude estimate of the amount of information (measured in decimal digits[2]) that is contained in these numbers. Let us introduce $d(n) := \log_{10}(n+1)$ to measure the number of *effective* decimal digits of a positive integer n (the ceiling of $d(n)$ is the number of digits needed to write n in decimal notation); we also define $d(n_1, \ldots, n_k) := d(n_1) + \cdots + d(n_k)$, the number of effective digits in a sequence $n_1, \ldots, n_k$ of positive integers. The number of effective digits in the sequence of six numbers from which the angle β is constructed is $d(10, 21, 24, 9, 6, 16) \doteq 6.86$, which is more than two digits short of the 9.09 digits of precision. In this comparison we have not accounted for the information present in the structure of the construction; the extent of this information is hard to estimate, and it may well amount to more than two decimal digits.[3] Because of the uncertainty (and vagueness) of our estimates we cannot tell whether the accuracy of the construction is in any sense 'surprisingly good' or not.

One nice thing that can be said about the construction of the angle β is that anybody who can count to 24 can do it. Let us compare this to the simple-minded graph-paper construction that approximates $\tan(40°)$ by a fraction m/n of positive integers and is at least as accurate as the construction of β. Let $\varepsilon := |\beta - 40°| = |\varepsilon_{40}|$, $t_{\text{lo}} := \tan(40° - \varepsilon)$ ($t_{\text{lo}} = t_{40}$), and $t_{\text{hi}} := \tan(40° + \varepsilon)$. The simplest fraction lying in the closed interval $[t_{\text{lo}} \mathinner{..} t_{\text{hi}}]$ is $s_{40} = 48797/58154$,[4] and the error is $\delta_{40} := \arctan(s_{40}) - 40° = 0.000\,101''$. The approximation $\arctan(s_{40})$ of 40° is precise to $-\log_{10}\bigl(|\delta_{40}|/40°\bigr) \doteq 9.15$ digits, while the number of effective digits in the fraction s_{40} is $d(48797) + d(58154) \doteq 9.45$,

[2] Instead of the usual binary digits, better known as "bits".

[3] I have only very vague ideas of how to *define* the information content of the structure of geometric constructions, and each of these ideas leads to absurdly large values. By the way, I feel that already the fact that we base our construction on six numbers somehow accounts for $\log_{10} 6 \doteq 0.78$ decimal digits, before the construction proper even starts.

[4] Let $0 < x < y$. The simplest fraction m/n in the closed interval $[x \mathinner{..} y]$ has n the smallest positive integer for which there is at least one fraction k/n in the interval, while $m = \lceil nx \rceil$. The fraction m/n is the simplest in the sense that if m_1/n_1 is any fraction in the interval, then $n_1 \geqslant n$ and $m_1 \geqslant m$. When $n > 1$, the fraction m/n is the only fraction of the form k/n in the interval. (To prove the last assertion, consider the Farey series of order n, consisting of all fractions p/q, $0 \leqslant p \leqslant q \leqslant n$, $\gcd(p, q) = 1$.)

so in this case we can safely claim that s_{40} is an underachiever as an approximation. Moreover, anyone actually carrying out the construction of the angle $\arctan(s_{40})$ would have a great deal of counting to do.

Let us glance back, at the graph-paper construction of the angle $\frac{1}{2}\alpha$ (Figure 5). If we do not double $\frac{1}{2}\alpha$, but instead use it to approximate 40° by $60° - \frac{1}{2}\alpha$, then the approximation is precise to 4.93 digits, while the number of effective digits in numbers used by the construction is $d(3,4,3,3) = 2.51$ digits. With the ballpark estimate of the information contained in the structure of the construction as being worth 1.5 digits, the construction is still 'under-informed' by almost a whole digit, so the precision of this construction may be considered as 'moderately surprising'.

In our second (and last) attempt at finding a graph-paper construction approximating one of the angles $\varphi = 40°, 20°, 10°, 5°$, we seek the approximation of $t = \tan(\varphi)$ in the form

$$t = \frac{p}{k + \sigma\sqrt{m^2+n^2} + \tau\sqrt{q^2+r^2}} ,$$

where p, m, n, q, r are positive integers no larger than some prescribed upper bound M, k is an integer with $|k| \leqslant M$, while σ, $\tau = \pm 1$. As before, we choose $M = 25$; this time we get the most accurate approximation for angle $\varphi = 10°$,

$$t_{10} = \frac{6}{-24 + \sqrt{15^2+22^2} + \sqrt{19^2+25^2}} = \frac{6}{-24 + \sqrt{709} + \sqrt{986}} ,$$

and the error is $\varepsilon_{10} := \arctan(t_{10}) - 10° \doteq 0.000\,073''$. The corresponding graph-paper

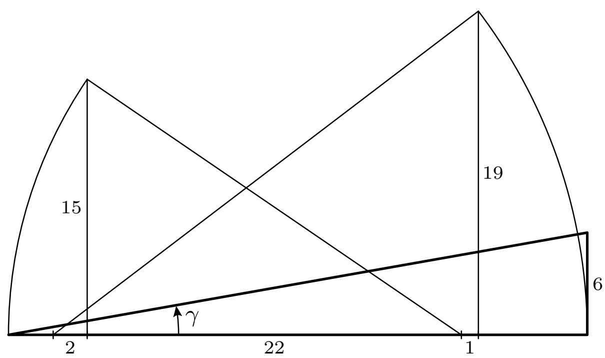


Figure 7: An approximate graph-paper construction of the 10° angle.

construction of the angle $\gamma = \arctan(t_{10})$ is shown in Figure 7.

To conclude this essay, we allow ourselves a little digression. Ian Stewart gives in his *Galois Theory* (third edition) the approximate construction for $\sqrt{\pi}$,[5] due to Ramanujan; it is shown in Figure 8. The construction runs as follows. Let AB be the diameter of a circle with the center O. Bisect AO at M, trisect OB at T. Draw the perpendicular to AB at T, meeting the circle at P. Draw |BQ| = |TP|, and join AQ. Draw OS, TR parallel to BQ. Draw |AD| = |AS|, and |AC| = |RS| with AC tangential to the circle

[5] Squaring the circle somehow goes hand-in-hand with the angle trisection, and in particular, with the construction of the regular nonagon. And no, we won't try to double the cube.

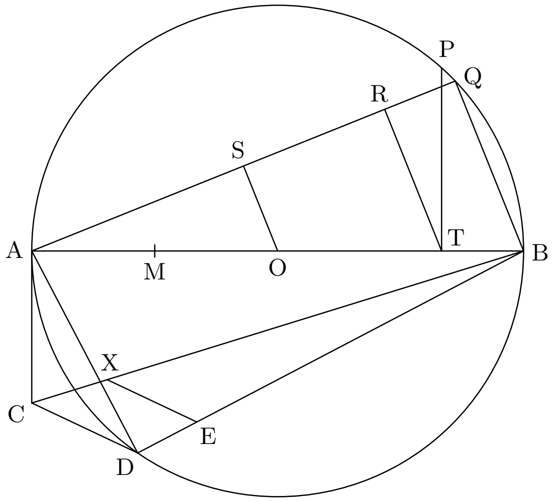


Figure 8: Srinivasa Ramanujan's approximate squaring of the circle: $|BX| : |OB| = \sqrt{355/113} \doteq \sqrt{\pi}$.

at A. Join BC, BD, CD. Draw $|BE| = |BM|$, draw EX parallel to CD. Then the square on BX has approximately the same area as the circle.

In the Ramanujan's construction the ratio $|BX| : |OB| = \sqrt{355/113}$ appears as if by magic; there are no numbers in the construction (if we are prepared to ignore the bisection and the trisection), the ratio is conjured up out of the construction's structure. Let us concoct, for comparison, a graph-paper construction of the same ratio. Noting

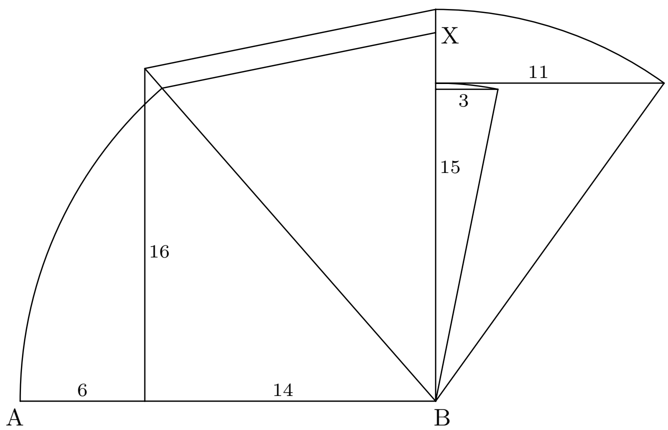


Figure 9: An almost-graph-paper construction of the ratio $|BX| : \frac{1}{2}|AB| = \sqrt{355/113}$.

that $113 = 7^2 + 8^2$ and $355 = 3^2 + 11^2 + 15^2$, we arrive at the construction shown in Figure 9, which is quite transparently building the required ratio from the numbers $2 \cdot 7$, $2 \cdot 8$, 3, 11, and 15. (The last step of the construction, the one that determines the point X, requires drawing the parallel to a line which is askew — neither horizontal nor vertical — so we have here only an almost-graph-paper construction.)

And here is another construction[6] of $\sqrt{355/113}$ (this one is not on graph paper).

[6]It is the last one, I promise.

Noting that

$$\sqrt{\frac{355}{113}} = \sqrt{3 + \frac{\left(\frac{1}{2}\right)^2}{1+\left(\frac{7}{8}\right)^2}}\,,$$

we geometrically compute the expression on the right hand side as in the left panel of Figure 10: A, B, ..., F are vertices of a regular hexagon, G is the midpoint of EF,

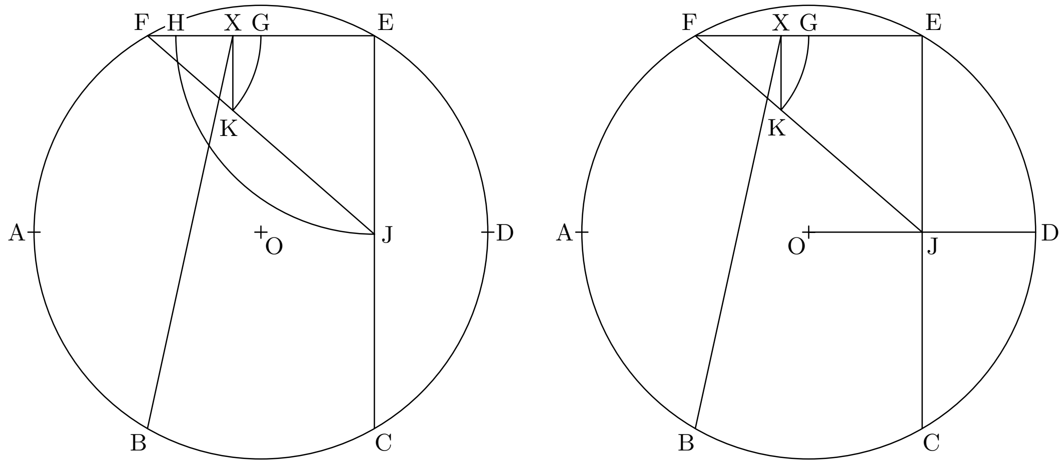


Figure 10: Another construction of the ratio $|BX| : |OA| = \sqrt{355/113}$ (left panel), and a construction of the ratio $|BX| : |OA| = \sqrt{22/7}$ (right panel).

$|FH| = \frac{1}{4}|FG|$, $|EJ| = |EH|$, $|FK| = |FG|$, and KX is parallel to CE; we have then $|BX| : |OA| = \sqrt{355/113}$. The point J lies below and very close to the midpoint of OD; if we move J to the midpoint of OD, we obtain a simpler construction, shown in the right panel of Figure 10, of the ratio $\sqrt{22/7}$.

How this construction came about? Well, I got this bright idea of rewriting 355/113 as $3 + 16/113$, and since I already knew that $113 = 7^2 + 8^2$, the rest naturally followed. I very much doubt that the construction is original; almost surely I only re-discovered it. MathWorld article “Circle squaring” has this to say about approximate circle squarings that produce the approximation 355/113 of π (or the approximation $\sqrt{355/113}$ od $\sqrt{\pi}$):

> Ramanujan (1913–1914), Olds (1963), Gardner (1966, pp. 92–93), and Bold (1982, p. 45) give geometric constructions for $355/133 = 3.1415929\ldots$.

The references are below. Gardner’s book is currently unavailable (I ordered it, in the hope that it will eventually reappear), while the rest are out of print or otherwise out of my reach.

References:

Bold, B. “The Problem of Squaring the Circle”. Ch. 6 in *Famous Problems of Geometry and How to Solve Them.* New York: Dover, pp. 39–48, 1982.

Gardner, M. “The Transcendental Number Pi”. Ch. 8 in Martin Gardner’s *New Mathematical Diversions: More Puzzles, Problems, Games, and Other Mathematical Diversions.* New York: Simon and Schuster, pp. 91–102, 1966.

Olds, C. D. *Continued Fractions.* New York: Random House, pp. 59–60, 1963.

Ramanujan, S. “Modular Equations and Approximations to π”. Quart. J. Pure. Appl. Math. **45**, 350–372, 1913–1914.