The join-induction principle for closure operators on dcpos

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1 Preliminaries

In this section we present notation and terminology, and state some basic facts.

If X and Y are sets, A is a subset of X, and F is a set of maps $X \to Y$, then we write $F(A)$ for the set $\{f(a) \mid f \in F \text{ and } a \in A\}$. We write $\{f\}(A)$ as $f(A)$, and $F\big(\{a\}\big)$ as $F(a)$.

Let X be a set and F a set of maps $X \to X$. A **fixed point** of F is an element $x \in X$ such that $f(x) = x$ for every $f \in F$. We denote the set of all fixed points of F by $\mathrm{Fix}(F)$ and call it the **fixed point set** of F; if F consists of a single map f, we write $\mathrm{Fix}(\{f\})$ as $\mathrm{Fix}(f)$. Note that $\mathrm{Fix}(\varnothing) = X$. We say that a subset A of X is **closed under** F if $F(A) \subseteq A$ (that is, if $f(a) \in A$ for every $a \in A$ and every $f \in F$); thus, in particular, $x \in X$ is a fixed point of F if and only if $\{x\}$ is closed under F.

Let P be a poset.

For any a, $b \in P$ we write variuos intervals (closed, open, half-open) from a to b as[1]

$$[a\,..\,b] \;=\; \{x \in P \mid a \leqslant x \leqslant b\}\,, \qquad (a\,..\,b) \;=\; \{x \in P \mid a < x < b\}\,,$$
$$[a\,..\,b) \;=\; \{x \in P \mid a \leqslant x < b\}\,, \qquad (a\,..\,b] \;=\; \{x \in P \mid a < x \leqslant b\}\,.$$

We also use notation, for any $a \in P$,

$$[a\,..) \;=\; \{x \in P \mid a \leqslant x\}\,, \qquad (a\,..) \;=\; \{x \in P \mid a < x\}\,,$$
$$(..\,a] \;=\; \{x \in P \mid x \leqslant a\}\,, \qquad (..\,a) \;=\; \{x \in P \mid x < a\}\,,$$

for intervals that have no given upper or lower bound.[2]

If x, $y \in P$ are distinct and $[x\,..\,y] = \{x, y\}$, then we say that x **is covered by** y, or that y **covers** x, and write $x \lessdot y$ (or $y \gtrdot x$). The least reflexive and transitive relation $\leqslant'$ on the set P that contains the relation $\lessdot$ (i.e. the reflexive transitive hull of $\lessdot$) is a partial ordering on the set P, and it is contained in the partial ordering $\leqslant$. If $\leqslant'$ coincides with $\leqslant$, then we say that the covering relation $\lessdot$ is the **Hasse diagram** of the poset P, and also say that P **has a Hasse diagram**.

[1] I have been using the dot-dot notation for intervals ever since I saw it (and fell in love with it) in *Concrete Mathematics* [1]. Originally this notation was suggested by C. A. R. Hoare and Lyle Ramshaw. The dot-dot notation is preferable to the comma notation $[-,-]$, $(-,-)$ etc, which is quite overloaded already without coercing it to serve also for intervals.

[2] Such intervals are 'boundless' in one direction, which is not the same as unbounded; for example, the interval $[a\,..)$ is bounded from above if and only if it has a largest element u, and in that case $[a\,..) = [a\,..\,u]$.

If P is not empty and any two $x, y \in P$ have an upper bound in P, then P is said to be **(upward) directed**. Equivalently, P is directed if every finite subset of P has an upper bound in P; in this formulation, a directed poset is seen to be nonempty because the empty set has an upper bound in it. A subset D of a poset P is said to be **directed** if the subposet D of P is a directed poset.

Let $x \in P$ and $A \subseteq P$. We write $x \uparrow A = \{a \in A \mid x \leqslant a\} = [x\,..) \cap A$. We write $x \geqslant A$ ($x \leqslant A$) to mean that $x \geqslant a$ ($x \leqslant a$) for every $a \in A$, i.e. that x is an upper (lower) bound of A. Similarly, we write $x > A$ ($x < A$) to mean that $x > a$ ($x < a$) for every $a \in A$, and say that x is a **strict upper (lower) bound** of A.

A mapping $f\colon P \to Q$ between posets is said to be **order preserving**, or **increasing**[3], if $x \leqslant y$ in P always implies $f(x) \leqslant f(y)$ in Q, and is said to be **order reversing**, or **decreasing**, if $x \leqslant y$ in P always implies $f(x) \geqslant f(y)$ in Q. A composite of two increasing maps is of course an increasing map, etc.

Let P be a poset, and denote by N the set of all increasing maps $P \to P$. Then N is a monoid for composition of maps. The pointwise ordering $\leqslant$ of maps (i.e. given any $f, g \in N$, $f \leqslant g$ means that $f(x) \leqslant g(x)$ for every $x \in P$) defines a partial order on N which agrees with composition: if $f, f', g \in N$, then $f \leqslant f'$ implies $fg \leqslant f'g$ by definition of $\leqslant$ in N,[4] and it also implies $gf \leqslant gf'$ because g is increasing.[5] It follows that for any $f, f', g, g' \in N$, relations $f \leqslant f'$ and $g \leqslant g'$ imply $ff' \leqslant gg'$.

Let P be a poset, and let f be an endomap on P.

We say that f **ascends** on $x \in P$ if $x \leqslant f(x)$. We say that f is **ascending**[6] if f ascends on every element of P. A composite of two ascending maps is an ascending map. Dually, we say that f **descends** on $x \in P$ if $x \geqslant f(x)$, and we say that f is descending if it descends on every element of P.

We call f a **closure operator** on P if it is ascending, increasing, and idempotent. If f is a closure operator, then $f(P) = \mathrm{Fix}(f)$ (because f is idempotent) and for any $x \in P$, $f(x)$ is the least fixed point of f above x, i.e. the least element of $x \uparrow \mathrm{Fix}(f)$. A subset C of P with the property that for every $x \in P$ the set $x \uparrow C$ has a least element is called a **closure system** on P. If C is a closure system on P, and for every $x \in P$ we denote the least element of the set $x \uparrow C$ by $f(x)$, then f is a closure operator on P and $\mathrm{Fix}(f) = C$. We denote by $\mathrm{Cl}(P)$ the set of all closure operators on P partially ordered by

[3] A.k.a. 'monotone' or 'monotone increasing', also 'isotone'. A certain terminological school, which delights in naming notions according to absence of certain properties ("non-negative" is typical), insists on calling such a mapping 'nondecreasing'. Let us analyze this term. First, that "non" cannot mean "not", i.e. 'nondecreasing' is not 'not decreasing'. The proper understanding of the "non" is "nowhere": let us say that a mapping $f\colon P \to Q$ between posets decreases on a pair $x < y$ in P if $f(x) > f(y)$ in Q; then call the mapping f 'nondecreasing' if it decreases on no pair $x < y$ in P. Now if Q is totally (i.e. linearly) ordered, then f is 'nondecreasing' if and only if $x < y$ in P always implies $f(x) \leqslant f(y)$ in Q, and this is precisely the intended meaning of 'nondecreasing'. Good... However, for a general poset Q, the mapping f is 'nondecreasing' if and only if for any pair $x < y$ in P we have $f(x) \not> f(y)$ in Q. This is not what we want; we want $x < y$ to imply $f(x) \leqslant f(y)$. To achieve this, we have to invent another notion. Let us say that f 'discomparizes' a pair $x < y$ if $f(x)$ and $f(y)$ are not comparable in Q, and call f 'nondiscomparizing' if it does not discomparize any pair $x < y$ in P. Then the desired notion is 'nondecreasing-and-nondiscomparizing'. Now this term may have a pleasant marypoppinsian flavour to it, but my tastes are simple, so I prefer to call order preserving mappings just 'increasing'.

[4] Thus this implication holds for any maps $f, f', g\colon P \to P$.

[5] And *this* implication holds for any maps $f, f'\colon P \to P$ and an increasing map $g\colon P \to P$.

[6] A.k.a. 'extensive', 'expansive', 'inflating', or (confusingly) 'increasing'.

pointwise ordering, and by $\mathrm{CS}(P)$ the set of all closure systems on P partially ordered by inclusion. The mapping $\mathrm{Cl}(P) \to \mathrm{CS}(P) : f \mapsto \mathrm{Fix}(f)$ is an antiisomorphism of posets.

We say that f is an **interior operator** on P if f is a closure operator on P^{op}, and we say that $J \subseteq P$ is an **interior system** on P if J is a closure system on P^{op}.

If f is ascending and increasing, we call it a **preclosure map**[7] on P. The set M of all preclosure maps on P is a submonoid of the monoid N of all increasing maps on P; M consists of all those $f \in N$ that satisfy the inequality $\mathrm{id}_P \leqslant f$. Every submonoid H of M is a directed subset of M relative to the pointwise ordering: H is not empty, and if $f, g \in H$, then $fg \in H$, where $f \leqslant fg$ because g is ascending and f is increasing, and $g \leqslant fg$ because f is ascending.

Closure operators on P are the idempotent members of the monoid M. If f is a closure operator on P and g is a preclosure map on P, then the inequality $g \leqslant f$ is equivalent to either of the equalities $gf = f$, $fg = f$.

Suppose f is a closure operator on P. If a subset S of $f(P)$ has a join in the subposet $f(P)$ of P, we write it as $\bigvee^f S$. For any subset S of P which has a join $\bigvee S$ in P, the set $f(S)$ has a join $\bigvee^f f(S) = f\big(\bigvee S\big)$ in $f(P)$. In particular, if $S \subseteq f(P)$ has a join $\bigvee S$ in P, then $S = f(S)$ has the join $\bigvee^f S = f\big(\bigvee S\big)$ in $f(P)$. Caution: a subset of $f(P)$ may have a join in $f(P)$ without having a join in P. Let $f'\colon P \to f(P)$ be the mapping f with the codomain restricted to the subposet $f(P)$ of P; then f' preserves all joins that exist in P.

Now let f be a preclosure map on P. If a set S of fixed points of f (i.e. $S \subseteq \mathrm{Fix}(f)$) has a meet $\bigwedge S$ in P, then $\bigwedge S$ is a fixed point of f; that is, $\mathrm{Fix}(f)$ is closed under all meets that exist in P.[8] If P has a largest element $\top$, then $\top$, the empty meet[9], is a fixed point of f. Supposing f is a closure operator, let S be any subset of $\mathrm{Fix}(f) = f(P)$; if S has a meet in the subposet $f(P)$ of P, then this meet is also the meet of S in P. Thus for a closure operator f, we need not distinguish between a meet of $S \subseteq f(P)$ in P and its meet in $f(P)$: if one of the meets exists, then the other one exists, too, and the two are equal.

Let L be a complete lattice.

Let f be closure operator on L. Since in L all joins and meets exist, the closure system $f(L)$, as a subposet of L, is a complete lattice: for any subset S of $f(L)$, the meet of S in $f(L)$ is the meet $\bigwedge S$ taken in L, while the join of S in $f(L)$ is $\bigvee^f S = f\big(\bigvee S\big)$ with the join $\bigvee S$ taken in L. For any $S \subseteq L$ we have $f\big(\bigvee S\big) = f\big(\bigvee f(S)\big) = \bigvee^f f(S)$; the restriction $f'\colon L \to f(L)$ of f preserves all joins. The closure system $f(L)$ is closed under arbitrary meets. Conversely, if $C \subseteq L$ is closed under arbitrary meets, then for every $x \in L$ the set $x \uparrow C$ has a least element $\bigwedge(x \uparrow C)$, thus C is a closure system.

The poset $\mathrm{CS}(L)$ is a closure system on the complete lattice $\mathscr{P}L$ (partially ordered by inclusion), and is therefore a complete lattice. Since the poset $\mathrm{Cl}(L)$ is antiisomorphic to the poset $\mathrm{CS}(L)$, $\mathrm{Cl}(L)$ is also a complete lattice. In both complete lattices, $\mathrm{CS}(L)$ and $\mathrm{Cl}(L)$, meets are easily described: meets in $\mathrm{CS}(L)$ are intersections, while meets in $\mathrm{Cl}(L)$ are computed pointwise. Joins are another matter entirely. For each $S \subseteq L$ we denote by $\mathrm{cs}(S)$

[7] A.k.a. 'inflationary' map, which may be confused with a merely 'inflating' map. I am here trying to coin the term "preclosure map", for the following reason. Closure operators have it easy: after a single ascending step they sit down and then stay there forever, contemplating their navels. We can regard a preclosure map as aspiring to closure — it ever climbs up, up, up… striving to reach a resting place. In general this is an unattainable goal (think of $n \mapsto n + 1$ in $\mathbb{N}$). However, preclosure maps living in certain special posets (dcpos) can always fulfill their aspirations.

[8] For a stronger version of this closure property, see Lemma 13 in Section 2.

[9] By an empty meet (join) we always mean the meet (join) of the empty subset of a poset P: the empty meet is the largest element of P (if it exists), and the empty join is the least element of P (if it exists).

the least closure system on L containing S; it is easy to see that $\mathrm{cs}(S) = \{\bigwedge T \mid T \subseteq S\}$. Then for any $\mathcal{C} \subseteq \mathrm{CS}(L)$ the join of $\mathcal{C}$ in $\mathrm{CS}(L)$ is $\mathrm{cs}\big(\bigcup \mathcal{C}\big)$. For any set G of closure operators on L, the join of G in $\mathrm{Cl}(L)$ is the closure operator h such that $\mathrm{Fix}(h) = \mathrm{Fix}(G)$. But this description of the join h is not quite satisfactory; given G and an $x \in L$, how can we reach $h(x)$, the least fixed point of G above x?

Let P and Q be posets. A **(covariant) Galois connection** $(f, g) : P \rightleftharpoons Q$ from the poset P to the poset Q is a pair of maps $f\colon P \to Q$ and $g\colon Q \to P$ such that

$$f(x) \leqslant y \iff x \leqslant g(y) \qquad \text{for all } x \in P \text{ and all } y \in Q\,. \tag{1}$$

Note that (f, g) is a Galois connection $P \rightleftharpoons Q$ if and only if (g, f) is a Galois connection $Q^{\mathrm{op}} \rightleftharpoons P^{\mathrm{op}}$; thus there is a certain upside-down symmetry, or *duality*, between the roles of the maps f and g (and the posets P and Q) in a Galois connection.

The following proposition spells out basic properties of Galois connections.

Proposition 1. *Let $(f, g) : P \rightleftharpoons Q$ be a Galois connection from a poset P to a poset Q. Then:*

(i) *f and g are increasing;*

(ii) *$fgf = f$ and $gfg = g$;*

(iii) *gf is a closure operator on P, while fg is an interior operator on Q;*

(iv) *$gf(P) = g(Q)$ and $fg(Q) = f(P)$;*

(v) *if $f'\colon g(Q) \to f(P)$ is the restriction of f and $g'\colon f(P) \to g(Q)$ is the restriction of g, then f' is an isomorphism of subposets, and g' is its inverse.*

(vi) *f preserves all joins that exist in P: if $S \subseteq P$ has a join $\bigvee S$ in P, then $f(S)$ has a join $f\big(\bigvee S\big)$ in Q; dually, g preserves all meets that exist in Q;*

(vii) *g is surjective $\iff$ $gf = \mathrm{id}_P$ $\iff$ f is injective;*
f is surjective $\iff$ $fg = \mathrm{id}_Q$ $\iff$ g is injective.

Having a Galois connection $(f, g) : P \rightleftharpoons Q$, we say that the map f is a **left adjoint** to the map g, and that g is a **right adjoint** to f, and write $f \dashv g$; also, we say that the closure operator gf on P is the **unit**, and that the interior operator fg on Q is the **counit**, of the Galois connection.

Proposition 2. *Suppose we have posets P and Q and maps $f\colon P \to Q$ and $g\colon Q \to P$. If f and g are increasing, gf is ascending, and fg is descending, then $(f, g) : P \rightleftharpoons Q$.*

Suppose that we have a map $f\colon P \to Q$ between posets; when precisely is f a left adjoint of some map $g\colon Q \to P$?

Proposition 3. *Let $f\colon P \to Q$ be a map between posets. The map f is a left adjoint of some map $g\colon Q \to P$ if and only if it is increasing and satisfies the condition that for every $y \in Q$ the set $G(y) = \{x \in P \mid f(x) \leqslant y\}$ has a greatest element. When a right adjoint g to f exists, it is unique.*

Dually, we have the following criterion for existence of a left adjoint:

Proposition 4. *Let $g\colon Q \to P$ be a map between posets. The map g is a right adjoint of some map $f\colon P \to Q$ if and only if it is increasing and satisfies the condition that for every $x \in P$ the set $F(x) = \{y \in Q \mid x \leqslant g(y)\}$ has a least element. When a left adjoint f to g exists, it is unique.*

A **contravariant Galois connection** from a poset P to a poset Q is defined as a covariant Galois connection $P \rightleftharpoons Q^{\mathrm{op}}$. Now (f, g) is a contravariant Galois connection from P to Q if and only if (g, f) is a contravariant Galois connection from Q to P, so there is a complete symmetry between the roles of the maps f and g (and the posets P and Q). For example, both maps constituting a contravariant Galois connection $(f, g) : P \rightleftharpoons Q^{\mathrm{op}}$, regarded as maps between P and Q (*not* Q^{op}), are decreasing, both composites gf and fg are closure operators (on P resp. Q), and the restrictions $f'\colon g(Q) \to f(P)$ of f and $g'\colon f(P) \to g(Q)$ of g are antiisomorphisms, inverse to each other, of subpposets $g(Q)$ of P and $f(P)$ of Q.

If (f, g) is a Galois connection from a complete lattice L to a complete lattice M, then f preserves all joins and, dually, g preserves all meets. A kind of converse also holds.

Proposition 5. *Let L and M be complete lattices. A map $f\colon L \to M$ has a right adjoint if and only if it preserves all joins, and dually, a map $g\colon M \to L$ has a left adjoint if and only if it preserves all meets.*

Let A be a set; the powerset $\mathscr{P}A$, ordered by inclusion, is a complete lattice. If f is a closure operator on the complete lattice $\mathscr{P}A$,[10] then the closure system $f(\mathscr{P}A)$ on $\mathscr{P}A$ is closed under arbitrary[11] intersections, and this property characterizes closure systems on $\mathscr{P}A$. We have already met an example of a closure system on a powerset: the set of all closure systems on a complete lattice L is a closure system on $\mathscr{P}L$.

Consider a Galois connection $(f, g) : \mathscr{P}A \rightleftharpoons \mathscr{P}B$. The defining property of a Galois connection is in this case

$$f(X) \subseteq Y \iff X \subseteq g(Y) \qquad \text{for all } X \subseteq A \text{ and all } Y \subseteq B\,.$$

Since the map f preserves all unions, it is completely given by its values $f\bigl(\{x\}\bigr)$ on one-element sets $\{x\} \subseteq A$; but f uniquely determines its right adjoint g, so we also know g if we know sets $f\bigl(\{x\}\bigr)$:

$$f(X) \;=\; \bigcup_{x\in X} f\bigl(\{x\}\bigr)\,, \qquad g(Y) \;=\; \bigl\{x \in X \bigm| f\bigl(\{x\}\bigr) \subseteq Y\bigr\}\,, \qquad \text{all } X \subseteq A, \text{ all } Y \subseteq B\,.$$

The sets $f\bigl(\{x\}\bigr) \subseteq B$, $x \in X$, can be chosen arbitrarily, since the map $A \to \mathscr{P}B : x \mapsto f\bigl(\{x\}\bigr)$ can be always uniquely extended to a map $\mathscr{P}A \to \mathscr{P}B : X \mapsto f(X)$ that preserves all unions, by setting $f(X) = \bigcup_{x\in X} f\bigl(\{x\}\bigr)$ for any $X \subseteq A$. A map $\rho\colon A \to \mathscr{P}B$ is essentially the same thing as a relation $R \subseteq A\times B$: given ρ, we define $x \mathrel{R} y :\iff \rho(x) \ni y$,

[10] Very often f is said to be a closure operator on a set A. This cannot cause any confusion when working exclusively with closure operators on powersets. However, we will always refer to f as a closure operator on $\mathscr{P}A$, since we will soon find ourselves in a situation where we will have closure operators on a poset P and at the same time also some closure operator on $\mathscr{P}P$. Ditto for closure systems on $\mathscr{P}A$.

[11] Since here we are working in a complete lattice $\mathscr{P}A$, the 'intersection' of the empty set of subsets of A is A itself, the top element of $\mathscr{P}A$.

and given R, we set $\rho(x) := \{y \in B \mid x\ R\ y\}$. If we associate in this way a relation R with the mapping $x \mapsto f(\{x\})$ for the map f in our Galois connection, then the maps f and g can be expressed in terms of the relation R:

$$f(X) = \{y \in B \mid \exists x \in X\colon x\ R\ y\} =: R_*(X), \qquad \text{for } X \subseteq A$$
$$g(Y) = \{x \in A \mid \forall y \in B\colon x\ R\ y \Longrightarrow y \in Y\} =: R_\dagger(Y), \qquad \text{for } Y \subseteq B.$$

The relation R is arbitrary: any $R \subseteq A \times B$ gives us $(R_*, R_\dagger) : \mathscr{P}A \rightleftharpoons \mathscr{P}B$.

Suppose we have a *contravariant* Galois connection $(f, g) : \mathscr{P}A \rightleftharpoons (\mathscr{P}B)^{\mathrm{op}}$. In this case both maps f and g convert arbitrary unions to intersections, and we have

$$f(X) = \bigcap_{x \in X} f(\{x\}), \qquad g(Y) = \bigcap_{y \in Y} g(\{y\}), \qquad \text{all } X \subseteq A, \text{ all } Y \subseteq B.$$

Introducing the relation $R \subseteq A \times B$ defined by $x\ R\ y :\Longleftrightarrow f(\{x\}) \ni y \Longleftrightarrow x \in g(\{y\})$, the maps f and g are expressed by the relation R as

$$f(X) = \{y \in B \mid \forall x \in X\colon x\ R\ y\} =: R^*(X), \qquad \text{for } X \subseteq A$$
$$g(Y) = \{x \in A \mid \forall y \in Y\colon x\ R\ y\} = (R^{\mathrm{op}})^*(Y), \qquad \text{for } Y \subseteq B.$$

The relation R is arbitrary: any $R \subseteq A \times B$ gives us $\big(R^*, (R^{\mathrm{op}})^*\big) \colon \mathscr{P}A \rightleftharpoons (\mathscr{P}B)^{\mathrm{op}}$.

Let A be a set. We now introduce closure rules of the form "if B is present, that c is also present", where $B \subseteq A$ and $c \in A$, and describe the way in which sets of such closure rules determine closure systems on $\mathscr{P}A$ and their corresponding closure operators on $\mathscr{P}A$.

A **closure rule** on A is a pair $(B, c) \in (\mathscr{P}A) \times A$, which we shall write as $B \mapsto c$; the set B is called the **body**, and the element c is called the **head**, of the closure rule $B \mapsto c$. We shall denote by $\mathrm{CR}(A)$ the set $(\mathscr{P}A) \times A$ of all closure rules on A. Given any set $R \subseteq \mathrm{CR}(A)$ of closure rules on A, we shall write $R\colon B \mapsto c$ to mean that the closure rule $B \mapsto c$ belongs to R. Sometimes a set R of closure rules will be known and fixed through a part of discussion; in such a case we will write simply $B \mapsto c$ instead of $R\colon B \mapsto c$; this abuse of notation will not cause any confusion.

A **nullary closure rule** is one of the form $\varnothing \mapsto c$ and shall be written $\mapsto c$. A **unary closure rule** is one of the form $\{b\} \mapsto c$, usually written $b \mapsto c$.

For any subset X of A and any closure rule $B \mapsto c$ on A we say that X **obeys** $B \mapsto c$, or that $B \mapsto c$ **is obeyed** by X, if $B \subseteq X$ implies $c \in X$.

The relation "obeys" between the sets $\mathscr{P}A$ and $\mathrm{CR}(A)$ gives rise to the contravariant Galois connection $(\varrho, \sigma) : \mathscr{P}\mathscr{P}A \rightleftharpoons \mathscr{P}\mathrm{CR}(A)$, where for each set $\mathcal{X}$ of subsets of A, $\rho(\mathcal{X})$ is the set of all closure rules on A obeyed by every $X \in \mathcal{X}$, and for each set R of closure rules on A, $\sigma(R)$ is the set of all subsets of A that obey every closure rule in R.

Proposition 6. *Let A be a set.*

A set $\mathcal{X}$ of subsets of A is of the form $\sigma(R)$ for some set R of closure rules on A if and only if it is closed under arbitrary intersections.

A set R of closure rules on A is of the form $\varrho(\mathcal{X})$ for some set $\mathcal{X}$ of subsets of A if and only if it satisfies the following two conditions:

- *reflexivity: $R\colon B \mapsto b$ for any $b \in B \subseteq A$;*
- *transitivity: for any $B, C \subseteq A$ and $d \in A$, if $R\colon B \mapsto c$ for every $c \in C$, and $C \mapsto d$, then $B \mapsto d$.*

A reflexive and transitive set of closure rules on A is said to be a **closure theory** on A. If f is a closure operator on $\mathscr{P}A$, the corresponding closure theory consists of all closure rules $B \mapsto c$ on A with $f(B) \ni c$. Conversely, if R is a closure theory on A, the corresponding closure operator on $\mathscr{P}A$ maps $X \subseteq A$ to $\{y \in A \mid R\colon X \mapsto y\}$. This correspondence between closure operators on $\mathscr{P}A$ and closure theories on A is an isomorphism of complete lattices.

Let R be a set of closure rules on A. The closure theory $\overline{R} = \varrho\sigma(R)$ is the least reflexive and transitive set of closure rules on A that contains R, i.e. it is the reflexive transitive hull of R. Any closure rule in $\overline{R}$ is said to be **derived** from R. A closure rule $X \mapsto y$ on A is derived from R if and only if there exists a mapping χ from some subset Z of A to ordinal numbers so that

- ⋄ for every $z \in Z$, either $z \in X$, or $R\colon B \mapsto z$ for some $B \subseteq Z$ with $\chi(B) < \chi(z)$, or both,
- ⋄ $y \in Z$;

and no, we do not need the axiom of choice to prove this.

2 Closure operators on dcpos

A **directed complete partial order**, or a **dcpo**, is a poset in which all directed joins exist. Let P be a dcpo, and let M be the pointwise ordered composition monoid of all preclosure maps on P. In M all directed joins exist, and they are computed pointwise: if F is a directed subset of M, then at each $x \in P$ the set $F(x)$ is directed, thus the map $\varphi\colon x \mapsto \bigvee F(x)$ is well defined, and one easily verifies that it is a preclosure map; it follows that $\varphi = \bigvee F$ in the poset M. Recall that every submonoid of M is a directed subset of M.

The following proposition describes how a set of preclosure maps on a dcpo generates a closure operator.

Proposition 7. *Let P be a dcpo, and let G be a set of preclosure maps on P. Then* $\mathrm{Fix}(G)$ *is a closure system on P, and the corresponding closure operator $\overline{G}$ on P with* $\mathrm{Fix}(\overline{G}) = \mathrm{Fix}(G)$ *is the least of all closure operators on P that are $\geqslant G$. Moreover, the following 'join-induction principle' holds: if a subset of P is closed under G and under directed joins in P, then it is closed under $\overline{G}$.*

Proof. In the pointwise ordered composition monoid M of all preclosure maps on P, let H be the intersection of all submonoids that contain G and are closed under directed joins in M; H is the least such submonoid. Since H is a directed subset of M, the join $h = \bigvee H$ in M exists. Then $h \in H$, because H is closed under directed joins, thus h is the largest element of H. Since H is a submonoid of M, we have $hh \in H$, hence $hh \leqslant h$, which shows that h is a closure operator. We have $h \geqslant G$ because H contains G. Let $d \geqslant G$ be a closure operator; then $d \in M$. The set $D = \{f \in M \mid f \leqslant d\}$ is a submonoid of M since $\mathrm{id}_P \in D$, and since $f,\, f' \in M$ and $f,\, f' \leqslant d$ imply $ff' \leqslant dd = d$; D is clearly closed under directed joins and contains G, thus it contains H, whence $h \leqslant d$.

Now let A be a subset of P that is closed under G and under directed joins. Let F be the set of all $f \in M$ such that $f(A) \subseteq A$. Then F contains G; F is evidently a submonoid of M, and it is closed under directed joins in M, because A is closed under directed joins in P and because the directed joins in M are computed pointwise. It follows that $H \subseteq F$, hence $h \in F$, that is, $h(A) \subseteq A$.

For any $g \in G$ we have $gh = h$ because g is a preclosure map, h is a closure operator, and $g \leqslant h$; this means that every element of $h(P) = \mathrm{Fix}(h)$ is a fixed point of G. Conversely, if a is a fixed point of G, then the set $\{a\}$ is closed under G, and since it is evidently closed under directed joins in P, it is closed under h, thus $h(a) = a \in \mathrm{Fix}(h)$.

We see that the closure operator $\overline{G} = h$ has the properties stated in the proposition. ☐

Let P, G, and $\overline{G}$ be as in Proposition 7. We shall say that the closure operator $\overline{G}$ is **generated by** the set G of preclosure maps.

The join-induction principle mentioned in Proposition 7 will be usually used in the following manner: in a dcpo P, if $x \in A$, where $A \subseteq P$ is closed under G and under directed joins, then $\overline{G}(x) \in A$.

In a dcpo, the join-induction principle characterizes the least one among the closure operators that are above a set of preclosure maps.

Proposition 8. *Let P be a dcpo, let G be a set of preclosure maps on P, and let $d \geqslant G$ be a closure operator on P. Suppose that d satisfies the following condition: any subset of P that is closed under G and under directed joins is closed under d. Then $d = \overline{G}$.*

Proof. We know from Proposition 7 that $d \geqslant \overline{G}$. To prove the opposite inequality, let $a \in P$ and let A be the interval $\big(..\overline{G}(a)\big]$ in P. The set A contains a and is evidently closed under directed joins; it is also closed under G, since for every $x \in A$ and every $g \in G$ we have $g(x) \leqslant g\big(\overline{G}(a)\big) = \overline{G}(a)$. Since d satisfies the join-induction principle stated in the proposition, it follows that A is closed under d; in particular $d(a) \in A$, and hence $d(a) \leqslant \overline{G}(a)$. ☐

Let c be a closure operator on a dcpo P. The subposet $\mathrm{Fix}(c) = c(P)$ of P is a dcpo: if $S \subseteq c(P)$ is directed in $c(P)$, it is also directed in P and it has a join $\bigvee S$ in P; then $\bigvee^c S = c\big(\bigvee S\big)$ is the join of S in $c(P)$. The restriction $c' \colon P \to c(P)$ of the closure operator c preserves all directed joins: if $S \subseteq P$ is directed, then $c'(S) = c(S)$ is directed, and $c'\big(\bigvee S\big) = c\big(\bigvee S\big) = c\big(\bigvee c(S)\big) = \bigvee^c c'(S)$.

Tarski's fixed point theorem, a version for dcpos, easily follows from Proposition 7.

Corollary 9. *Let f be an increasing endomap on a dcpo P. Then the set of all fixed points of f, as a subposet of P, is a dcpo, and for every $x \in P$ on which f ascends there exists a least fixed point of f above x. If P has a least element, then* $\mathrm{Fix}(f)$ *has a least element, i.e. f has a least fixed point.*

Proof. Let $A = \{x \in P \mid x \leqslant f(x)\}$; clearly A contains all fixed points of f. Since $x \leqslant f(x)$ implies $f(x) \leqslant f\big(f(x)\big)$, the set A is closed under f. If $S \subseteq A$ is directed, then $s \leqslant f(s) \leqslant f\big(\bigvee S\big)$ for every $s \in S$, hence $\bigvee S \leqslant f\big(\bigvee S\big)$, so A is closed under directed joins. Thus the subposet A is a dcpo[12], and the restriction $g \colon A \to A$ of f is a preclosure map on A. The closure operator $\overline{g}$ on A maps each $x \in A$ to the least element above x in the set $\mathrm{Fix}(\overline{g}) = \mathrm{Fix}(g) = \mathrm{Fix}(f)$, and this set, as a subposet of A, and hence of P, is a dcpo[13]. If P has a least element $\bot$, then $\bot \in A$, and $\overline{g}(\bot)$ is the least fixed point of f. ☐

[12] We can say that the subposet A is a sub-dcpo of P, since directed joins in A are computed in P.

[13] The subposet $\mathrm{Fix}(f)$ of P is *not* a sub-dcpo of P, in general: the join in $\mathrm{Fix}(f)$ of a directed $S \subseteq \mathrm{Fix}(f)$ is $\overline{g}(\bigvee S)$, where the join $\bigvee S$ is taken in A (and hence in P).

The last statement in Corollary 9 is the bare-bones Tarski's fixed point theorem for dcpos; let us restate it on its own. We say that a dcpo is **pointed** if it has a least element.

Corollary 10. *Every increasing endomap on a pointed dcpo has a least fixed point.*

The bare-bones Tarski's fixed point property actually characterizes pointed dcpos:

Proposition 11. *A poset, on which every increasing endomap has a least fixed point, is a pointed dcpo.*

We omit the proof, which uses the axiom of choice, and that with abandon.

The full Tarski's fixed point theorem for dcpo's can be derived without much ado from its bare-bones version (while ignoring Proposition 7, of course). Here is how. Assume that we know Corollary 10, and let f be an increasing endomap on a dcpo P. We introduce the set $A = \{x \in P \mid x \leqslant f(x)\}$, which happens to contain all fixed points of f, and then show that A is a sub-dcpo of P (i.e. that it is closed under directed joins in P) and that it is closed under f, precisely as in the proof of Corollary 9. For each $a \in A$, the subposet $[a\,..)$ of P is a pointed sub-dcpo of P which is closed under f; the restriction of f to an endomap on $[a\,..)$ is increasing, hence has a least fixed point, write it $g(a)$, which is the least fixed point of f above a. This means that $\mathrm{Fix}(f)$ is a closure system on A, while g is the corresponding closure operator on A. Since A is a dcpo, $g(A) = \mathrm{Fix}(f)$ is also a dcpo.

As a consequence of Proposition 7, if P is a dcpo, then in the poset $\mathrm{Cl}(P)$ every subset has a join, i.e. $\mathrm{Cl}(P)$ is a complete lattice.

Corollary 12. *The poset* $\mathrm{Cl}(P)$ *of all closure operators on a dcpo* P *is a complete lattice, and so is the poset* $\mathrm{CS}(P)$ *of all closure systems on* P*. In* $\mathrm{Cl}(P)$*, the join of a set* G *of closure operators is the closure operator* $\overline{G}$ *generated by* G*, while in* $\mathrm{CS}(P)$*, the meet of a set* $\mathcal{C}$ *of closure systems is the intersection* $\bigcap \mathcal{C}$*.*

Proof. Let $G \subseteq \mathrm{Cl}(P)$; by its definition, $\overline{G}$ is evidently the join of G in $\mathrm{Cl}(G)$. Since $g \mapsto \mathrm{Fix}(g)$ is an antiisomorphism $\mathrm{Cl}(P) \to \mathrm{CS}(P)$, and $\mathrm{Fix}\big(\overline{G}\big) = \mathrm{Fix}(G) = \bigcap_{g \in G} \mathrm{Fix}(g)$ for any $G \subseteq \mathrm{Cl}(P)$, it follows that meets of closure systems are intersections. □

The complete lattice $\mathrm{CS}(P)$ of all closure systems on a dcpo P is a closure system on $\mathscr{P}P$. Can we exhibit a set of closure rules that determines the closure system $\mathrm{CS}(P)$? The set of all closure rules obeyed by the closure system $\mathrm{CS}(P)$ is easy to describe: it consists of all closure rules $B \mapstochar\rightarrow c$ with $B \subseteq P$ and $c \in P$, where c is fixed by every closure operator on P that fixes every element of B; here we can replace "every closure operator" by "every preclosure map", since $\mathrm{Fix}(f) = \mathrm{Fix}\big(\overline{f}\big)$ for any preclosure map f. But this is not what we desire: we want a set of closure rules that can be described in terms of the structure of the dcpo P itself.

For a while, let P be any poset.

We shall say that $B \mapsto c$ is a **default closure rule** associated with P if $c \in P$ and $B \subseteq (c..)$ and c is the only lower bound of B in $[c..)$. We shall denote the set of all default closure rules associated with P by $\mathrm{R}_{\mathrm{df}}(P)$, and shall write a default closure rule $B \mapsto c$ as $B \mapsto_{\mathrm{df}} c$; by the agreed abuse of notation, we shall also write $B \mapsto_{\mathrm{df}} c$ to mean $\mathrm{R}_{\mathrm{df}}(P)\colon B \mapsto c$. Note that whenever $B \mapsto_{\mathrm{df}} c$ and $B \subseteq B' \subseteq (c..)$, then also $B' \mapsto_{\mathrm{df}} c$. In particular, if c appears as the head of some default closure rule, then $(c..) \mapsto_{\mathrm{df}} c$, and the converse is clearly also true. That is, c is a head of some default closure rule if and only if the set $(c..)$ does not have a least element; equivalently, c is *not* a head of any default closure rule if and only if there exists $d > c$ with $[d..) = (c..)$, and in such a case the element d is unique and covers c. Note also that $\mapsto_{\mathrm{df}} c$ if and only if c is a maximal element of P.

Lemma 13. *For any preclosure map f on a poset P, $\mathrm{Fix}(f)$ obeys $\mathrm{R}_{\mathrm{df}}(P)$.*

Proof. Let $B \mapsto_{\mathrm{df}} c$ with $B \subseteq \mathrm{Fix}(f)$. For any $b \in B$ we have $c \leqslant f(c) \leqslant f(b) = b$, thus $f(c) \in [c..)$ is a lower bound of B and hence $f(c) = c \in \mathrm{Fix}(f)$. □

Let $\mathcal{K}$ be the set of the fixed point sets of all closure operators on the poset P, let $\mathcal{G}$ be the set of the fixed point sets of all preclosure maps on P, and let $\mathcal{D}$ be the closure system on $\mathscr{P}P$ determined by the set of all default closure rules associated with P. Clearly $\mathcal{K} \subseteq \mathcal{G}$, and we have just shown that $\mathcal{G} \subseteq \mathcal{D}$. Here is an example of a poset P for which both inclusions are proper. Take $P = \mathbb{N} \cup \{a\}$ with $a \notin \mathbb{N}$, where the subset $\mathbb{N}$ is ordered in the usual way, and there is an additional relationship $0 < a$; the poset P has a Hasse diagram which is a *very* long hockey stick. Then $\mathcal{K}$ is the set of all infinite subsets of P that contain the set $\{0, a\}$, $\mathcal{G}$ is the set of all subsets of P that contain $\{0, a\}$, and $\mathcal{D} = \mathcal{G} \cup \{\{a\}\}$.

In the preceding example the set $\mathcal{K}$ is not closed under finite intersections, while the set $\mathcal{G}$ is closed under arbitrary intersections. It is easy to see that $\mathcal{G}$ is always, for any poset P, closed under *finite* intersections. Indeed, $P = \mathrm{Fix}(\mathrm{id}_P)$, while if f and g are any preclosure maps, then fg is a preclosure map, and $\mathrm{Fix}(fg) = \mathrm{Fix}(f) \cap \mathrm{Fix}(g)$ simply because f and g are ascending: if x is a common fixed point of f and g, then it is a fixed point of fg; conversely, if x is a fixed point of fg, then $x \leqslant g(x) \leqslant f(g(x)) = x$, thus $g(x) = x$, whence also $f(x) = f(g(x)) = x$. Perhaps $\mathcal{G}$ is always closed under arbitrary intersections? Right now I am unable to come up with a counterexample, so this is (for me, at least) an open question.

If the poset P is a dcpo, then we know from Proposition 7 that $\mathcal{K} = \mathcal{G}$; moreover, in this case also $\mathcal{G} = \mathcal{D}$, that is, the subsets of P obeying the default closure rules associated with P are precisely the closure systems on P. The proof that we give below uses the axiom of choice.

Proposition 14. *Let P be a dcpo. Then the default closure rules associated with P determine the closure system $\mathrm{CS}(P)$ on $\mathscr{P}P$.*

Proof. We already know that every closure system on P satisfies $\mathrm{R}_{\mathrm{df}}(P)$. Conversely, suppose that $A \subseteq P$ is not a closure system on P. Then there is $b \in P$ for which the set $b \uparrow A$ does not have a least element (possibly because it is empty). In the nonempty partially ordered subset of P consisting of all $x \geqslant b$ with $x \uparrow A = b \uparrow A$ there exists a maximal chain U, which is not empty. Then $u = \bigvee U$ exists, $u \geqslant b$, $u \uparrow A = b \uparrow A$, and $u \notin A$ since otherwise u would be the least element of $b \uparrow A$; also for every $x > u$ (where possibly there is no such x) we have $x \uparrow A \neq b \uparrow A$, by maximality of U. We claim that $(b \uparrow A) \mapsto_{\mathrm{df}} u$: we have $b \uparrow A = u \uparrow A \subseteq (u\,..)$, and if $x \in (u\,..)$, then $x \uparrow (b \uparrow A) = x \uparrow A$ is a proper subset of $b \uparrow A$ and hence x is not a lower bound of $b \uparrow A$. We have found a default closure rule $(b \uparrow A) \mapsto_{\mathrm{df}} u$ which A fails to satisfy. □

The use of the axiom of choice in the proof of Proposition 14 cannot be eliminated.

Proposition 15. *In absence of the axiom of choice, the statement of the proposition 14 is equivalent to the axiom of choice.*

Proof. We already know that the axiom of choice implies the validity of Proposition 14. Conversely, assume that Proposition 14 is true. Let P be any nonempty dcpo. Since $\mathrm{R}_{\mathrm{df}}(P)$ determines $\mathrm{CS}(P)$, and since all closure systems on P are nonempty, there must exist a nullary default closure rule associated with P, which means that P has a maximal element. Now let Q be any poset. The set of all chains in Q (the empty chain included), partially ordered by inclusion, is closed under directed unions; it is therefore a nonempty dcpo, and by what we have just proved there exists a maximal chain in Q. □

By the way, in the course of the above proof we also proved the following directed form of Zorn's lemma: in absence of the axiom of choice, the statement that every nonempty dcpo has a maximal element is equivalent to the axiom of choice.

3 The frame of nuclei on a preframe

A **preframe**[14] is a dcpo that is also a meet-semilattice, in which finite meets distribute over directed joins, that is, in which the following directed distributive law holds:

$$x \wedge \bigvee Y \;=\; \bigvee_{y \in Y} (x \wedge y)\,, \qquad \text{any } x \in P, \text{ any directed } Y \subseteq P\,.$$

(If $Y \subseteq P$ is directed, then $\{x \wedge y \mid y \in Y\}$ is also directed.) A preframe has all finite meets, including the empty one, which is the top element $\top$ of the preframe. A preframe is said to be **compact**, if its top element is inaccessible by directed joins, i.e. if $\top$ cannot be obtained as the join of a directed subset that does not contain $\top$, or equivalently, if every directed subset whose join is $\top$ already contains $\top$.

We will consider structures hat are slightly more general than preframes, namely dcpos with binary meets (equivalently, with nonempty finite meets) that satisfy the directed distributive law; we shall call such a structure a **preframe′**.[15] If we add to a preframe′ P a top element $\top \notin P$ (even if P already has a top element), we get a compact preframe $P \cup \{\top\}$; conversely, if P is a compact preframe with a top element $\top$, then $P \smallsetminus \{\top\}$ is a preframe′. In short, a preframe′ is just a 'beheaded' compact preframe.[16]

Let P be a preframe′. A closure operator on P that preserves nonempty finite meets is called a **nucleus** on P. A preclosure map on P that preserves nonempty finite meets is

[14] As defined in Escardó [2].

[15] Alternatively, we could call a 'preframe' what we now know as a preframe′, and would then say that a 'preframe' with a top element is a 'topped preframe', or a 'preframe$^\top$'.

[16] Which suggests the reading "chop!" for the prime symbol "′" in "preframe′".

called a **prenucleus** on P. A map $\gamma\colon P \to P$ is a prenucleus if it is ascending and preserves binary meets, and it is a nucleus if it is ascending and idempotent and preserves binary meets: since γ preserves binary meets, it is increasing and preserves nonempty finite meets. If γ is a prenucleus, and P happens to have a top element $\top$, then $\gamma(\top) = \top$ because γ is ascending; that is, if the empty meet in P exists, then γ preserves it.

For a little while, let P be any poset in which all binary meets exist. In the pointwise ordered set of *all* maps $P \to P$, any two maps γ, $\delta\colon P \to P$ have a meet $\gamma \wedge \delta$, which is computed pointwise, and the following is true:

(i) if γ and δ are ascending, so is $\gamma \wedge \delta$;

(ii) if γ and δ are increasing, so is $\gamma \wedge \delta$;

(iii) if γ and δ are closure operators, so is $\gamma \wedge \delta$;

(iv) if γ and δ preserve binary meets, so does $\gamma \wedge \delta$.

Properties (i) and (ii) are easily verified. For (iii), assume β and γ are closure operators on P and put $\beta = \gamma \wedge \delta$. Then β is increasing and ascending by (i) and (ii). Since $\beta\beta \leqslant \gamma\gamma = \gamma$, and similarly $\beta\beta \leqslant \delta$, we have $\beta\beta \leqslant \gamma \wedge \delta = \beta$, thus β is idempotent. Finally, to prove (iv), assume that γ and δ preserve binary meets and take any x, $y \in P$; then

$$\begin{aligned}(\gamma \wedge \delta)(x \wedge y) &= \gamma(x \wedge y) \wedge \delta(x \wedge y)\\ &= \gamma(x) \wedge \gamma(y) \wedge \delta(x) \wedge \delta(y)\\ &= (\gamma \wedge \delta)(x) \wedge (\gamma \wedge \delta)(y)\,.\end{aligned}$$

Because of (iii), in the poset $\mathrm{Cl}(P)$ all binary meets exists and they are computed pointwise.

Now let P be a preframe$'$. From what we have just shown it follows that the pointwise meet of two prenuclei on P is a prenucleus on P, and that the pointwise meet of two nuclei on P is a nucleus on P. Therefore, in the pointwise ordered set of all prenuclei on P all binary meets exist and they are computed pointwise, and the same is true for the pointwise ordered set of all nuclei on P.

How about the joins of sets of nuclei on a preframe$'$ P? They always exist, and they are computed in the complete lattice $\mathrm{Cl}(P)$ of all closure operators on P. We give a slightly more general result.

Proposition 16. *Let P be a preframe$'$, and let Γ be a set of prenuclei on P; then the closure operator $\overline{\Gamma}$, generated by the set Γ of preclosure maps, is a nucleus. In particular, if Γ consists of nuclei on P, then the join $\bigvee \Gamma$, taken in the complete lattice* $\mathrm{Cl}(P)$*, is a nucleus.*

Proof. Put $\delta = \overline{\Gamma}$. Let x, $y \in P$.

Since δ is increasing, we have $\delta(x \wedge y) \leqslant \delta(x) \wedge \delta(y)$; we must prove that the opposite inequality also holds.

We will first prove the weaker assertion $x \wedge \delta(y) \leqslant \delta(x \wedge y)$. Let A be the set of all $z \in P$ such that $x \wedge z \leqslant \delta(x \wedge y)$. The set A clearly contains y, and it is closed under directed joins by directed distributivity. For any $z \in A$ and any $\gamma \in \Gamma$ we have

$$x \wedge \gamma(z) \;\leqslant\; \gamma(x) \wedge \gamma(z) \;=\; \gamma(x \wedge z) \;\leqslant\; \gamma\big(\delta(x \wedge y)\big) \;=\; \delta(x \wedge y)\,,$$

hence $\gamma(z) \in A$; thus A is closed under Γ. The join-induction principle now gives $\delta(y) \in A$.

Since x and y above were arbitrary, we can substitute $\delta(x)$ for x in $x \wedge \delta(y) \leqslant \delta(x \wedge y)$ and get

$$\delta(x) \wedge \delta(y) \;\leqslant\; \delta\big(\delta(x) \wedge y\big) \;\leqslant\; \delta\big(\delta(x \wedge y)\big) \;=\; \delta(x \wedge y)\,,$$

where the second inequality holds because $\delta(x) \wedge y \leqslant \delta(x \wedge y)$. □

We shall denote by $\mathrm{N}(P)$ the pointwise ordered set of all nuclei on a preframe′ P.

Corollary 17. *For any preframe′ P, the poset $\mathrm{N}(P) \subseteq \mathrm{Cl}(P)$ is closed under arbitrary joins in the complete lattice $\mathrm{Cl}(P)$,*[17] *hence it is a complete lattice. Also, $\mathrm{N}(P)$ is closed under binary meets in $\mathrm{Cl}(P)$, since binary meets are computed pointwise in either one of the two complete lattices.*

Here we can see that our decision to consider preframes′ instead of only preframes was not so innocent as it seemed. For a preframe P it is trivial that there is a greatest nucleus on P, namely the constant map sending every element of P to the top element of P. In contrast, the existence of a greatest nucleus on a preframe′ is a quite nontrivial fact.

A **frame** is a complete lattice L in which finite meets distribute over arbitrary joins, which means that the following infinite distributivity law holds in L:

$$x \wedge \bigvee Y \;=\; \bigvee_{y \in Y} (x \wedge y) \qquad \text{for all } x \in L \text{ and all } Y \subseteq L\,.$$

As an ordered structure, a frame is the same thing as a complete Heyting algebra. The difference is in the structural features that are perceived as basic and have to be preserved by homomorphisms. For frames the basic operations are finite meets and arbitrary joins, and homomorphisms of frames are maps that preserve these operations; correspondingly, a subframe is a subset of a frame closed under finite meets and arbitrary joins, equipped with the frame structure induced from the 'ambient' frame.

Proposition 18. *For any preframe′ P, the complete lattice $\mathrm{N}(P)$ is a frame.*

Proof. Let $\beta \in \mathrm{N}(P)$ and $\Gamma \subseteq \mathrm{N}(P)$, and write $\delta = \bigvee \Gamma$, $\delta' = \bigvee_{\gamma \in \Gamma} (\beta \wedge \gamma)$ (joins taken in $\mathrm{N}(P)$, hence in $\mathrm{Cl}(P)$). We must show that $\beta \wedge \delta = \delta'$. The inequality $\beta \wedge \delta \geqslant \delta'$ holds because $\beta \wedge \delta \geqslant \beta \wedge \gamma$ for every $\gamma \in \Gamma$. To prove the other inequality, let $x \in P$ and put $A = \{z \in P \mid x \leqslant z \text{ and } \beta(x) \wedge z \leqslant \delta'(x)\}$. Clearly $x \in A$, and A is closed under directed joins by directed distributivity in P. To see that A is closed under Γ, take any $z \in A$ and any $\gamma \in \Gamma$; then $x \leqslant z \leqslant \gamma(z)$, and since $\beta(x) = \beta\beta(x)$, $\beta(x) \leqslant \beta(z)$, and $\beta(x) \leqslant \gamma\beta(x)$, we have

$$\begin{aligned} \beta(x) \wedge \gamma(z) \;&=\; \beta\beta(x) \wedge \beta(z) \wedge \gamma\beta(x) \wedge \gamma(z) \\ &=\; (\beta \wedge \gamma)\big(\beta(x) \wedge z\big) \leqslant\; (\beta \wedge \gamma)\big(\delta'(x)\big) \;=\; \delta'(x)\,, \end{aligned}$$

thus $\gamma(z) \in A$. By the join-induction principle it follows that $\delta(x) \in A$, that is, that $(\beta \wedge \delta)(x) = \beta(x) \wedge \delta(x) \leqslant \delta'(x)$. □

[17] That is, $\mathrm{N}(P)$ is a complete join-sub-semilattice of $\mathrm{Cl}(P)$. Or is it a complete sub-join-semilattice? Or, perhaps, a sub-(complete join-semilattice)?

Let P be a preframe′.

We shall call a subset of P a **nuclear system** on P if it is the fixed point set of some nucleus on P, and we shall denote by $\mathrm{NS}(P)$ the set of all nuclear systems on P, partially ordered by inclusion. If γ is any prenucleus on P, then $\mathrm{Fix}(\gamma) = \mathrm{Fix}(\overline{\gamma})$, where $\overline{\gamma}$ is the nucleus generated by the prenucleus γ; thus the fixed point set of any prenucleus on P is a nuclear system on P.

The mapping $\mathrm{N}(P) \to \mathrm{NS}(P) : \gamma \mapsto \mathrm{Fix}(\gamma)$ is an antiisomorphism of posets, so $\mathrm{NS}(P)$ is a complete lattice. The antiisomorphism $\mathrm{N}(P) \to \mathrm{NS}(P)$ is a restriction of the antiisomorphism $\mathrm{Cl}(P) \to \mathrm{CS}(P) : g \mapsto \mathrm{Fix}(g)$; since the joins in $\mathrm{N}(P)$ are taken in $\mathrm{Cl}(P)$, the meets in $\mathrm{NS}(P)$ are taken in $\mathrm{CS}(P)$, that is, $\mathrm{NS}(P)$ is a closure system on $\mathscr{P}P$ contained in the closure system $\mathrm{CS}(P)$ on $\mathscr{P}P$.

We know that the closure system $\mathrm{CS}(P)$ is determined by the default closure rules associated with P — provided we admit the axiom of choice, which we do, just for a little while. We shall exhibit some additional closure rules which, together with the default closure rules, determine the closure system $\mathrm{NS}(P)$.

Let $a,\, b \in P$. The set

$$(a \overset{*}{\Rightarrow} b) \;:=\; \{x \in P \mid x \wedge a \leqslant b\}$$

is nonempty (it contains b), it is a lower subset of P (clear), and it is closed under directed joins (by directed distributivity). In any dcpo, a lower subset that is closed under directed joins is called **Scott closed**; thus in our case, $(a \overset{*}{\Rightarrow} b)$ is a nonempty Scott closed subset of P; in particular, $(a \overset{*}{\Rightarrow} b)$ is a nonempty dcpo. Now define

$$(a \dot{\Rightarrow} b) \;:=\; \mathrm{Max}(a \overset{*}{\Rightarrow} b)\,,$$

i.e. $(a \dot{\Rightarrow} b)$ is the set of all maximal elements of $(a \overset{*}{\Rightarrow} b)$. By the axiom of choice, $(a \dot{\Rightarrow} b)$ is not empty; more, any $x \in (a \overset{*}{\Rightarrow} b)$ is below some $\dot{x} \in (a \dot{\Rightarrow} b)$ since $x \uparrow (a \overset{*}{\Rightarrow} b)$ is a nonempty dcpo with the least element x.

The following lemma does not require the axiom of choice.

Lemma 19. *Let P be a preframe′, and $C \subseteq P$ a nuclear system on P. If $a \in P$ and $c \in C$, then $(a \dot{\Rightarrow} c) \subseteq C$.*

Proof. Let γ be the nucleus on P whose fixed point set is C. Let $x \in (a \dot{\Rightarrow} c)$; we must show that x is a fixed point of γ. Since $x \wedge a \leqslant c$ and $\gamma(c) = c$, we have

$$\gamma(x) \wedge a \;\leqslant\; \gamma(x) \wedge \gamma(a) \;=\; \gamma(x \wedge a) \;\leqslant\; \gamma(c) \;=\; c\,,$$

thus $\gamma(x) \in (a \overset{*}{\Rightarrow} c)$; now $\gamma(x) \geqslant x$ and the maximality of x in $(a \overset{*}{\Rightarrow} c)$ imply $\gamma(x) = x$. □

However, the proof of the converse of the preceding lemma uses the axiom of choice.

Lemma 20. *Let P be a preframe′, and let $C \subseteq P$ be a closure system on P. If C satisfies the condition that for any $a \in P$ and any $c \in C$ the set $(a \dot{\Rightarrow} c)$ is contained in C, then C is a nuclear system on P.*

Proof. Let γ be the closure operator on P whose fixed point set is C. We must show that γ preserves binary meets. Take any $a,\, b \in P$; it suffices to prove that $a \wedge \gamma(b) \leqslant \gamma(a \wedge b)$. Since $a \wedge b \leqslant \gamma(a \wedge b)$, b lies in $\big(a \overset{*}{\Rightarrow} \gamma(a \wedge b)\big)$, and by the axiom of choice, $b \leqslant \dot{b}$ for some $\dot{b} \in \big(a \dot{\Rightarrow} \gamma(a \wedge b)\big)$. By our assumption about C, $\dot{b}$ is a fixed point of γ, thus $\gamma(b) \leqslant \gamma(\dot{b}) = \dot{b}$, and it follows that $a \wedge \gamma(b) \leqslant a \wedge \dot{b} \leqslant \gamma(a \wedge b)$. □

We shall say that a unary closure rule $b \mapsto c$ on a preframe′ P is a **nuclear closure rule** associated with P, and shall write the rule as $b \,\dot{\mapsto}\, c$, if $c \in (a \Rrightarrow b)$ for some $a \in P$; the set of all nuclear closure rules associated with P shall be denoted by $\mathrm{R}^{\cdot}(P)$.

We have arrived at the following result (albeit using the axiom of choice here and there):

Corollary 21. *Let P be a preframe′. Then the closure system $\mathrm{NS}(P)$ on $\mathscr{P}P$ is determined by the set $\mathrm{R}_{\mathrm{df}}(P) \cup \mathrm{R}^{\cdot}(P)$ of closure rules on P.*

4 The Hofmann–Mislove–Johnstone theorem

In this section we present a straightforward application of the join-induction principle in a proof of the Hofmann–Mislove–Johnstone theorem:

Theorem 22. *The compact fitted quotient frames of any frame are in order-reversing bijective correspondence[18] with the Scott open filters on the frame.*

If this sounds all Greek to you, do not panic; everything will be explained below — slowly and in sickening detail — before we embark on the actual proof of the theorem, which will be short and quite painless.

Frames we have already defined: a frame is a complete lattice in which finite meets distribute over arbitrary joins, and a frame homomorphism is a mapping from a frame to a frame that preserves finite meets and arbitrary joins.

From now on let L denote an arbitrary frame.

Consider a nucleus γ on L. The subposet $\gamma(L) = \mathrm{Fix}(\gamma)$ of L is a complete lattice in which arbitrary meets are computed in L and the join of any $S \subseteq \gamma(L)$ is $\bigvee^{\gamma} S = \gamma(\bigvee S)$. Moreover, the infinite distributivity law holds in the complete lattice $\gamma(L)$, so it is in fact a frame: given any $x \in \gamma(L)$ and any $Y \subseteq \gamma(L)$, we have $x \wedge \bigvee^{\gamma} Y = \gamma(x) \wedge \gamma(\bigvee Y) = \gamma(x \wedge \bigvee Y) = \gamma(\bigvee_{y \in Y}(x \wedge y)) = \bigvee^{\gamma}_{y \in Y}(x \wedge y)$. The restriction $\gamma' \colon L \to \gamma(L)$ of γ preserves finite meets because γ preserves them, and it preserves arbitrary joins because γ is a closure operator; thus γ' is a surjective frame homomorphism. This is why the nuclear system $\gamma(L)$ is also called a **quotient frame** of L.

Let $f \colon L \to K$ be any frame homomorphism. Since f preserves all joins, it has a right adjoint $g \colon K \to L$, which preserves all meets, thus the closure operator $\gamma = gf$ on L preserves finite meets, i.e. it is a nucleus on L. Now suppose that f is surjective. Denoting by $\gamma' \colon L \to \gamma(L)$ the restriction of γ and by $h \colon \gamma(L) \to K$ the restriction of f, we find that γ' is a surjective frame homomorphism and h is an isomorphism of frames such that $h\gamma' = f$; thus every surjective frame homomorphism from L is isomorphic to an 'inner' surjective frame homomorphism from L represented by a nucleus on L.

Another notion we have already defined is compactness of preframes, which applies also to frames (as very special preframes): a frame is compact if its top element is unreachable by directed joins. So we now know what is a compact quotient frame. 'Fitted' comes next.

[18] An 'order-reversing bijective correspondence' means an antiisomorphism of posets, that is, a bijection between posets such that both the bijection itself and its inverse are order-reversing.

We have mentioned that a frame is a complete Heyting algebra: for any two elements a, $b \in L$ there exists the **relative pseudo-complement** of a with respect to b, which is the unique element $(a \Rightarrow b) \in L$ such that

$$x \wedge a \leqslant b \iff x \leqslant (a \Rightarrow b) \qquad \text{for every } x \in L\,;$$

a moment's thought makes us see that $(a \Rightarrow b) = \bigvee\{x \mid x \wedge a \leqslant b\}$.

Let $a \in L$. The principal ideal $(..a]$ is a frame, the map $f_a \colon L \to (..a] : x \mapsto x \wedge a$ is a frame homomorphism, and the defining property of $(-\Rightarrow-)$ shows that the right adjoint of f_a is the map $g_a \colon (..a] \to L : y \mapsto (a \Rightarrow y)$; the nucleus $\mathrm{o}(a) = \mathrm{o}_a := g_a f_a$ on L maps $x \in L$ to $\mathrm{o}_a(x) = \big(a \Rightarrow (x \wedge a)\big) = (a \Rightarrow x)$.[19] The nucleus $\mathrm{o}(a)$ is called the **open nucleus** associated with a.

A nucleus γ on L, and the corresponding nuclear system $\gamma(L)$ on L, are said to be **fitted**, if γ is a join of open nuclei; we shall denote by $\mathrm{N_{fit}}(L)$ the set of all fitted nuclei on L, and by $\mathrm{NS_{fit}}(L)$ the set of all fitted nuclear systems (i.e. fitted quotient frames) on L. Subposet $\mathrm{N_{fit}}(L)$ of $\mathrm{N}(L)$ is a complete lattice because it is clearly closed under arbitrary joins in $\mathrm{N}(L)$.[20] Correspondingly, the subposet $\mathrm{NS_{fit}}(L)$ of $\mathrm{NS}(L)$ is a complete lattice; it is closed under arbitrary meets in $\mathrm{NS}(L)$, and since meets in $\mathrm{NS}(L)$ are intersections, $\mathrm{NS_{fit}}(L)$ is a closure system on $\mathscr{P}L$.

Below any nucleus $\gamma \in \mathrm{N}(L)$ there exists the largest fitted nucleus $\gamma^\phi \in \mathrm{N_{fit}}(L)$: γ^ϕ is simply the join of all open nuclei below γ. The mapping $\gamma \mapsto \gamma^\phi$ is an interior operator on $\mathrm{N}(L)$; it is fittingly called the **fitting** of nuclei on L.

Given a nucleus γ on L, which open nuclei on L are below γ? We claim that for any $a \in L$, the open nucleus $\mathrm{o}(a)$ is below γ if and only if $\gamma(a) = \top$. Indeed, if $\mathrm{o}(a) \leqslant \gamma$, then $\top = (a \Rightarrow a) = \mathrm{o}_a(a) \leqslant \gamma(a)$. For the opposite implication, first note that any endomap f on L that preserves binary meets, and is hence increasing, satisfies for any x, $y \in L$ the inequality $f(x \Rightarrow y) \leqslant \big(f(x) \Rightarrow f(y)\big)$, which follows from the inequality between the first and the last expressions in $f(x) \wedge f(x \Rightarrow y) = f\big(x \wedge (x \Rightarrow y)\big) \leqslant f(y)$. Now, if $\gamma(a) = \top$, then for any $x \in L$ we have $o_a(x) = (a \Rightarrow x) \leqslant \gamma(a \Rightarrow x) \leqslant \big(\gamma(a) \Rightarrow \gamma(x)\big) = \big(\top \Rightarrow \gamma(x)\big) = \gamma(x)$.

We can now write down an explicit formula for the fitting of a nucleus:

$$\gamma^\phi \;=\; \bigvee\{\mathrm{o}(a) \mid \gamma(a) = \top\} \qquad \text{for any } \gamma \in \mathrm{N}(P)\,.$$

So far we completely understand one side of the bijection mentioned in Theorem 22. There is not much left to understand on the other side. We have defined Scott closed subsets of a dcpo — they are lower subsets closed under directed joins. A subset of a dcpo is said to be **Scott open** if its complement in the dcpo is Scott closed. A subset U of a dcpo is therefore Scott open if and only if it is an upper subset unreachable by directed joins, where unreachability by directed joins means that for any directed subset S of the dcpo, $\bigvee S \in U$ implies $S \cap U \neq \varnothing$.

A subset V of a poset P is called a **filter** on P if it is a downward directed[21] upper subset of P. On our frame L, a filter is an upper subset closed under finite meets (including the empty meet, i.e. a filter always contains $\top$). Any filter V on L obeys the *modus ponens* rule: for all a, $b \in L$, if $a \in V$ and $(a \Rightarrow b) \in V$, then $b \in V$ (because $b \geqslant a \wedge (a \Rightarrow b) \in V$).

[19] We can verify directly that $o_a = (a \Rightarrow -)$ is a nucleus: it is ascending and increasing; it is idempotent: $\big(a \Rightarrow (a \Rightarrow x)\big) = \big((a \wedge a) \Rightarrow x\big) = (a \Rightarrow x)$; it preserves binary meets: $\big(a \Rightarrow (x \wedge y)\big) = (a \Rightarrow x) \wedge (a \Rightarrow y)$.

[20] Actually, $\mathrm{N_{fit}}(L)$ is a subframe of $\mathrm{N}(L)$, i.e. it is also closed under finite meets, since $\mathrm{o}(\bot) \colon x \mapsto \top$ is the largest nucleus, and $\mathrm{o}(a) \wedge \mathrm{o}(b) = \mathrm{o}(a \vee b)$ for any a, $b \in L$.

[21] I.e. V is (upward) directed in P^{op}.

For any nucleus γ on L, the set $\gamma^{-1}(\top)$ is a filter on L; we shall call filters of this form **nuclear filters** on L, and will denote by $\mathrm{NF}(L)$ the set of all nuclear filters on L partially ordered by inclusion. Since

$$\big(\bigwedge \Gamma\big)^{-1}(\top) \;=\; \bigcap\{\gamma^{-1}(\top) \mid \gamma \in \Gamma\} \qquad \text{for any } \Gamma \subseteq \mathrm{N}(L)\,,$$

we see that $\mathrm{NF}(L)$ is a closure system on $\mathscr{P}L$.

All is ready for the lemma relating compact quotient frames to Scott open nuclear filters. The proof given below is almost verbatim as in Escardó [2], and that proof is lifted from Johnstone [4]. Anyway, the lemma is an immediate consequence of the relationship between joins in a frame and joins in a quotient frame of the frame.

Lemma 23. *Let γ be a nucleus on a frame L. Then the quotient frame $\gamma(L)$ is compact if and only if the nuclear filter $\gamma^{-1}(\top)$ is Scott open.*

Proof. ($\Longrightarrow$) Suppose $\gamma(L)$ is compact, and let $S \subseteq L$ be directed with $\bigvee S \in \gamma^{-1}(\top)$. Since $\bigvee^{\gamma}\gamma(S) = \gamma\big(\bigvee S\big) = \top$ and $\gamma(L)$ is compact, there is some $s \in S$ with $\gamma(s) = \top$, i.e. with $s \in \gamma^{-1}(\top)$.

($\Longleftarrow$) Suppose $\gamma^{-1}(\top)$ is Scott open, and let $S \subseteq \gamma(L)$ be directed with $\bigvee^{\gamma} S = \top$. Since $\bigvee^{\gamma} S = \gamma\big(\bigvee S\big)$, we have $\bigvee S \in \gamma^{-1}(\top)$, and by Scot openness of $\gamma^{-1}(\top)$, there is some $s \in S$ with $s \in \gamma^{-1}(\top)$, that is, with $s = \gamma(s) = \top$. $\square$

The interior operator $\gamma \mapsto \gamma^{\phi}$ on $\mathrm{N}(L)$ is a counit of a Galois connection, which we now proceed to describe. For every $\gamma \in \mathrm{N}(L)$ put $\nabla\gamma := \gamma^{-1}(\top)$, and for every $S \in \mathscr{P}L$ put $\Delta S := \bigvee_{s\in S} \mathrm{o}(s)$. For any $\gamma \in \mathrm{N}(L)$ and any $S \in \mathscr{P}L$ the chain of equivalences

$$\begin{aligned} \Delta S \;\leqslant\; \gamma \;&\Longleftrightarrow\; \forall s{\in}S\colon \mathrm{o}(s) \leqslant \gamma \\ &\Longleftrightarrow\; \forall s{\in}S\colon \gamma(s) = \top \\ &\Longleftrightarrow\; S \subseteq \nabla\gamma \end{aligned}$$

shows that (Δ,∇) is a Galois connection $\mathscr{P}L \rightleftharpoons \mathrm{N}(L)$. By our definitions, $\Delta(\mathscr{P}L)$ is the set $\mathrm{N}_{\mathrm{fit}}(L)$ of all fitted nuclei on L, while $\nabla\mathrm{N}(L)$ is the set $\mathrm{NF}(L)$ of all nuclear filters on L. From the general properties of Galois connections it at once follows that $\Delta\nabla\colon \gamma \mapsto \gamma^{\phi}$ is an interior operator on $\mathrm{N}(L)$ and that for any nucleus γ, γ^{ϕ} is the largest fitted nucleus below γ (all of which we already know), while on the other side, $\mathrm{nf} := \nabla\Delta$ is a closure operator on $\mathscr{P}L$, where for any subset S of L, $\mathrm{nf}(S)$ is the least nuclear filter on L that contains S. We have the identities $\nabla\Delta\nabla = \nabla$ and $\Delta\nabla\Delta = \Delta$, which mean that $\big(\gamma^{\phi}\big)^{-1}(\top) = \gamma^{-1}(\top)$ for every $\gamma \in \mathrm{N}(L)$ and that $\bigvee_{s\in\mathrm{nf}(S)} \mathrm{o}(s) = \bigvee_{s\in S} \mathrm{o}(s)$ for every $S \subseteq L$. Moreover, restricting Δ to $\mathrm{NF}(L) \to \mathrm{N}_{\mathrm{fit}}(L)$ and ∇ to $\mathrm{N}_{\mathrm{fit}}(L) \to \mathrm{NF}(L)$, we get two isomorphisms of complete lattices that are inverses to each other.

At last, here comes the punch line — or should it be a punch lemma?

The following lemma is Lemma 4.3(2) in Escardó [2] — which in turn is Lemma 2.4(ii) in Johnstone [4] — with a spruced up proof.

Lemma 24. *Every Scott open filter on a frame L is nuclear.*

Proof. Let V be a Scott open filter on a frame L, and let $\gamma = \Delta V = \bigvee_{v \in V} \mathrm{o}(v)$. If $v \in V$, and $x \in L$ is such that $\mathrm{o}_v(x) = (v \Rightarrow x) \in V$, then $x \in V$ by modus ponens, which means that the set $L \smallsetminus V$ is closed under every open nucleus $\mathrm{o}(v)$ with $v \in V$. By assumption the set $L \smallsetminus V$ is closed under directed joins, and by the join-induction principle it follows that this set is closed under γ. But this means that $\gamma^{-1}(V) \subseteq V$, so certainly $\mathrm{nf}(V) = \gamma^{-1}(\top) \subseteq V$. Since also $V \subseteq \mathrm{nf}(V)$, it follows that $V = \mathrm{nf}(V)$ is a nuclear filter. □

After all the preparations, Theorem 22 is easy to prove.

Proof of Theorem 22. Denote by $\mathcal{Q}$ the set of all compact fitted quotient frames on L, and by $\mathcal{F}$ the set of all Scott open filters on L, with both sets partially ordered by inclusion. By Lemmas 23 and 24, the isomorphism $\varphi\colon \mathrm{NF}(P) \to \mathrm{NS}_{\mathrm{fit}}(P)^{\mathrm{op}} : V \mapsto \mathrm{Fix}(\Delta V)$ restricts to an isomorphism $\psi\colon \mathcal{F} \to \mathcal{Q}^{\mathrm{op}}$. □

Looking back at the proof of Lemma 24, we notice it does not work in a constructive set theory: we first pass from the filter V to its complement, then pass back from the complement to the filter, and this second passing is illegitimate in constructive logic. Lemma 4.3(2) in Escardó [2] is without blemish in this respect. As we shall see, we can repair this defect of the proof of Lemma 24, which will then become even simpler.

But first we must make clear what we mean when we say that a subset A of a dcpo P is unreachable by directed joins. If we are admitting classical logic, we can spell out this property of A in two equivalent ways: one is "for every directed subset S of P, if $s \notin A$ for every $s \in S$, then $\bigvee S \notin A$", and the other one is "for every directed subset S of P, if $\bigvee S \in A$, then $s \in A$ for some $s \in S$". However, if we constrain our reasonings to constructive logic, the two formulations above describe two distinct properties of A, where the second one implies the first one (i.e. it is the stronger of the two), but not conversely. *We hereby proclaim the second of the above formulations as the 'official' definition of unreachability of A by directed joins*, to be used in a constructive ambience. In particular, we use this definition when we define a Scott open subset of a dcpo as an upper subset unreachable by directed joins.

The tool we need to repair the proof of Lemma 24 is a version of the join-induction principle which is applicable to subsets of a dcpo unreachable by directed joins.

Let us say that a subset A of a set X is **inversely closed** under a set F of endomaps of X if $f^{-1}(A) \subseteq A$ (i.e. $f(x) \in A$ implies $x \in A$ for every $x \in X$) for every $f \in F$.

Proposition 7 (continued). *Also, the following alternative 'join-induction principle' holds: if a subset of P is inversely closed under G and unreachable by directed joins, then it is inversely closed under $\overline{G}$.*

Proof (continued). Suppose that $A \subseteq P$ is inversely closed under G and that it is unreachable by directed joins. Let F be the set of all $f \in M$ such that $f^{-1}(A) \subseteq A$; F contains G and it is a submonoid of M. Let E be a directed subset of F. Let $x \in P$, and suppose that $\left(\bigvee E\right)(x) = \bigvee E(x) \in A$; since A is unreachable by directed joins, there exists $e \in E$ with $e(x) \in A$, and we have $x \in A$ because A is inversely closed under e. It follows that A is inversely closed under $\bigvee E$. Thus F is closed under directed joins, so it contains H and with it the closure operator h, whence A is inversely closed under h. □

And here is the repaired proof of Lemma 24.

Constructive proof of Lemma 24. Let V be a Scott open filter on a frame L, and let $\gamma = \Delta V = \bigvee \mathrm{o}(V)$. If $v \in V$, and $x \in L$ is such that $\mathrm{o}_v(x) = (v \Rightarrow x) \in V$, then $x \in V$ by modus ponens, which means that V is inversely closed under $\mathrm{o}(V)$. By the join-induction principle for subsets unreachable by directed joins, V is inversely closed under γ, so certainly $\mathrm{nf}(V) = \gamma^{-1}(\top) \subseteq V$. Since also $V \subseteq \mathrm{nf}(V)$, we see that $V = \mathrm{nf}(V)$ is a nuclear filter. □

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