The Envelope-Folding Problem from Paul J. Nahin's "When Least is Best"

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The invitation, eagerly accepted

On the page 80 of "When Least is Best" the author Paul J. Nahin, after describing the envelope-folding problem, makes the following remark:

> This sounds like a simple question, but I don't think it is. If you don't agree, then shut the book right now and try your hand at it *before* you read what follows.

And that was what I did: I shut the book right then, and tried my hand at the problem. Below you'll find the story of my search for a solution.

Geometry of the envelope-folding problem

This is the envelope-folding problem, as formulated in the book:

> We are given a right triangle OAB, with perpendicular sides of lengths a and b meeting at the corner O, as shown in Figure 1. Suppose we fold the right angle over to place O at some point P on the hypotenuse. This can be done in infinity of ways. Each such way results in the folded triangle OYX having some area; our question is: what is the minimum possible area of OYX?

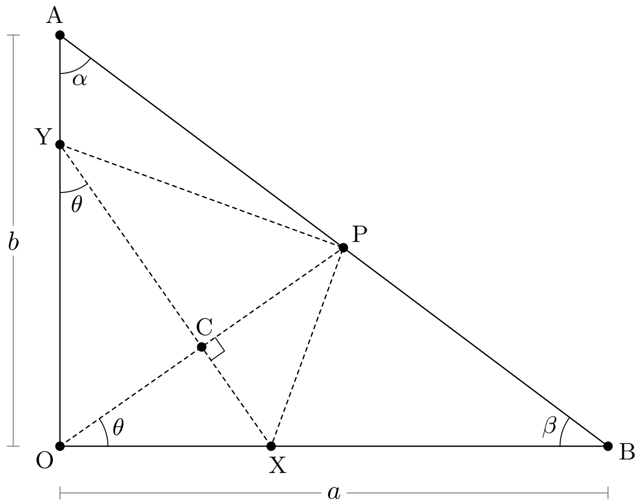

Figure 1: Geometry of the envelope-folding problem.

We denote the interior angle of the triangle OAB at the vertex A by α and the interior angle at the vertex B by β, thus $\alpha = \arctan(a/b)$, $\beta = \arctan(b/a)$, and $\alpha + \beta = \pi/2$.

Let C be the midpoint of the line segment OP. When we fold the right angle over, the line segment OC is carried onto the line segment PC. The crease created by the folding is the perpendicular bisector of the line segment OP; it intersects the side OB of the triangle at the point X, and the side OA at the point Y. After the folding, the triangle OYX is positioned as the triangle PYX.

The angle $\theta = \measuredangle\text{BOP}$ cannot vary through the full range $0 \leqslant \theta \leqslant \pi/2$, because of the constraints that the endpoint X of the crease must lie on the side OB of the triangle and that its other endpoint Y must lie on the side OA. The least possible value of θ is $\alpha/2$, when Y = A and AX bisects the interior angle at A, while the largest possible value is $\pi/2 - \beta/2 = \alpha/2 + \pi/4$, when X = B and BX bisects the interior angle at B (see Figure 2). When $0 \leqslant \theta < \alpha/2$ or $\alpha/2 + \pi/4 < \theta \leqslant \pi/2$, one endpoint of the crease lies on the hypotenuse, as in Figure 3. We therefore seek the minimum area of the triangle OYX with the angle θ restricted to the interval $\alpha/2 = \theta_{\text{lo}} \leqslant \theta \leqslant \theta_{\text{hi}} = \alpha/2 + \pi/4$.

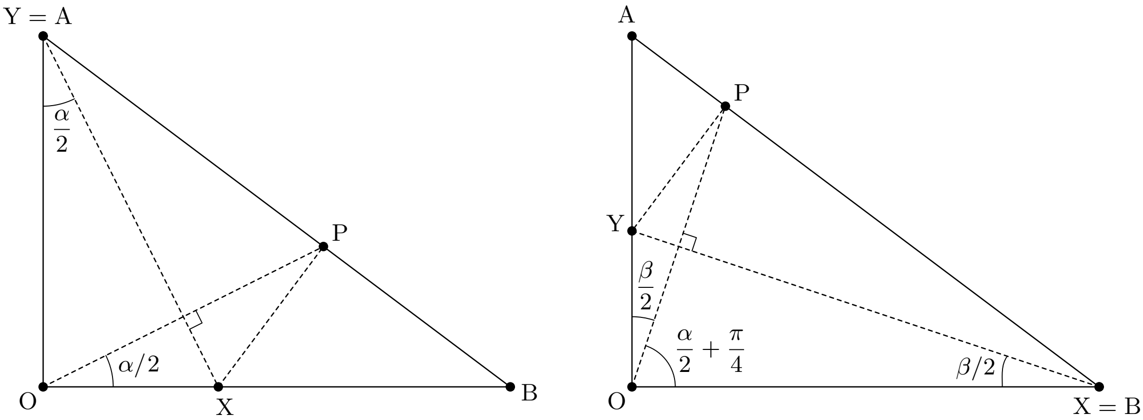

Figure 2: Foldings at $\theta = \theta_{\mathrm{lo}} = \alpha/2$ and $\theta = \theta_{\mathrm{hi}} = \alpha/2 + \pi/4$.

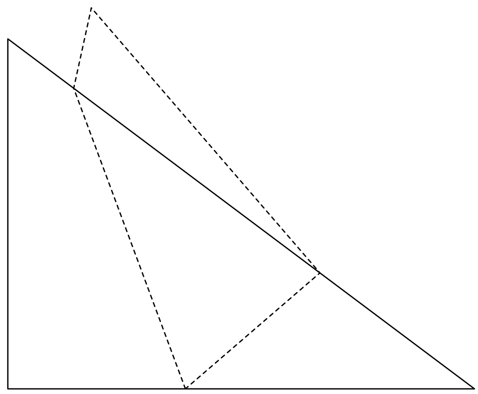

Figure 3: An illegal folding.

Let us set up an Euclidean coordinate system with the origin at O, the x-axis along OB, and the y-axis along OA. We choose $t = \tan(\theta)$ as the parameter that determines a folding, because the coordinates of the points P, C, X, and Y are rational functions of t, and so is the area of the triangle OYX. Our minimization problem has t restricted to the interval

$$\frac{a}{b + \sqrt{a^2 + b^2}} \;=\; \tan\left(\frac{\alpha}{2}\right) \;=\; t_{\mathrm{lo}} \;\leqslant\; t \;\leqslant\; t_{\mathrm{hi}} \;=\; \tan\left(\frac{\alpha}{2} + \frac{\pi}{4}\right) \;=\; \frac{a + \sqrt{a^2 + b^2}}{b}\,.$$

To get the coordinates of the point P, we intersect the line OP, equation $y = tx$, with the line AB, equation $x/a + y/b = 1$:

$$x_{\mathrm{P}} \;=\; \frac{ab}{at + b}\,, \qquad y_{\mathrm{P}} \;=\; \frac{abt}{at + b}$$

The coordinates of the point C are one half the coordinates of the point P: $x_{\mathrm{C}} = \frac{1}{2} x_{\mathrm{P}}$, $y_{\mathrm{C}} = \frac{1}{2} y_{\mathrm{P}}$. Next we compute the coordinates x_{X} and y_{Y} of the intersection points X and Y of the orthogonal bisector of the line segment OP, equation

$$y - y_{\mathrm{C}} \;=\; -\frac{1}{t}\big(x - x_{\mathrm{C}}\big)\,,$$

with the coordinate axes:

$$x_{\mathrm{X}} \;=\; \frac{1}{2} ab \frac{1 + t^2}{at + b}\,, \qquad y_{\mathrm{Y}} \;=\; \frac{x_{\mathrm{X}}}{t} \;=\; \frac{1}{2} ab \frac{1 + t^2}{t(at + b)}\,.$$

Finally, the area A of the triangle OYX is

$$A \;=\; \frac{1}{2} x_{\mathrm{X}} y_{\mathrm{Y}} \;=\; \frac{1}{8} a^2 b^2 \frac{(1 + t^2)^2}{t(at + b)^2}\,.$$

Seeking the minimum

At this point it is expedient to introduce the 'shape parameter' $w = b/a = \tan(\beta) = 1/\tan(\alpha)$ of the triangle OAB. Since we are assuming a non-degenerate triangle OAB, the shape parameter w is a strictly positive real number, $0 < w < +\infty$ (the values $w = 0$ and $w = +\infty$ correspond to the two opposite degenerate cases). If we so wish, we can further restrict w to the range $0 < w \leqslant 1$, because if $w > 1$ we simply swap the sides OA and OB of the triangle OAB, which changes w to $1/w < 1$; if the area of the triangle OYX attains the minimum at $\theta = \theta_{\text{opt}} = \theta_{\text{opt}}(w)$, hence at $t = t_{\text{opt}} = t_{\text{opt}}(w) = \tan\big(\theta_{\text{opt}}(w)\big)$, then $\theta_{\text{opt}}(w) = \pi/2 - \theta_{\text{opt}}(1/w)$ and $t_{\text{opt}}(w) = 1/t_{\text{opt}}(1/w)$.

With the use of the shape parameter w, the area of the triangle OYX can be expressed as

$$A \;=\; \frac{1}{8} b^2 \frac{(1+t^2)^2}{t(t+w)^2} \,.$$

We shall locate the value of t at which the area A attains the minimum by studying the stationary points of the function

$$f(t) \;=\; \frac{(1+t^2)^2}{t(t+w)^2}$$

of the real variable t.

The diagram of $f(t)$ (for $w = 3/4$) is shown in Figure 4. Staring a second or two at the formula for $f(t)$, we notice that $f(t) \to -\infty$ when $t \searrow -\infty$ or $t \nearrow -w$ or $t \searrow -w$ or $t \nearrow 0$, and that $f(t) \to +\infty$ when $t \searrow 0$ or $t \nearrow +\infty$ (just as the diagram suggests), therefore $f(t)$ attains an absolute maximum in each of the two intervals $t < -w$ and $-w < t < 0$ (where 'absolute' is meant relative to the part of the function on that interval), and it attains an absolute minimum in the interval $0 < t$. We conclude that $f(t)$ has at least three distinct stationary points.

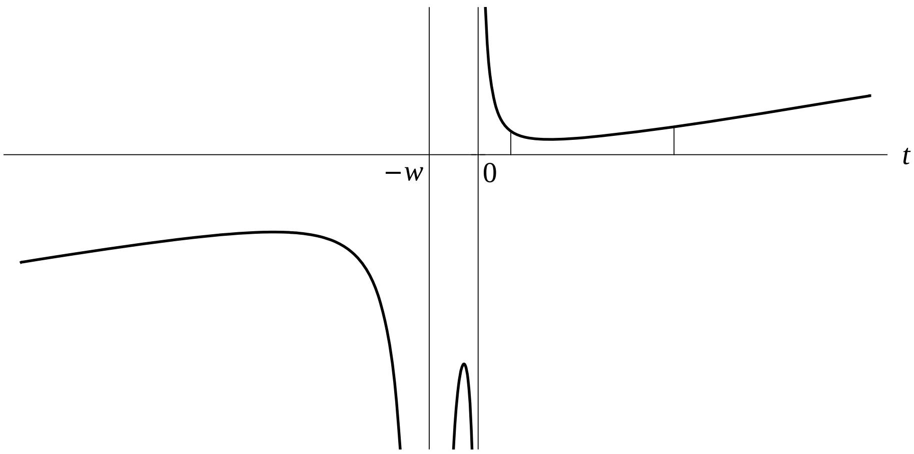

Figure 4: The diagram of $f(t)$, for $w = 3/4$. The marked interval is $t_{\text{lo}} \leqslant t \leqslant t_{\text{hi}}$.

The derivative of $f(t)$,

$$f'(t) \;=\; \frac{(1+t^2)(t^3 + 3wt^2 - 3t - w)}{t^2(t+w)^3} \,,$$

has at most three real zeros, namely the zeros of the cubic factor in the numerator. It follows that $f(t)$ has precisely three distinct real zeros, and so does the cubic factor. Let us denote the three zeros, in increasing order, by t_1, t_2, and t_3; then $t_1 < -w < t_2 < 0 < t_3$. The numbers t_1, $-w$, t_2, 0, and t_3 partition the t-axis into six open intervals (the leftmost and rightmost ones are unbounded); the signs of the derivative $f'(t)$ on these intervals are, left to right, $+, -, +, -, -, +$. In particular, we see that $f(t) > f(t_3)$ for $t > 0$, $t \neq t_3$.

Minimum found, with a twist

To determine t_1, t_2, and t_3, we must solve the cubic equation

$$t^3 + 3wt^2 - 3t - w \;=\; 0\,.$$

Since this equation has three distinct real roots, Cardano's formula is of no use; we have a specimen of *casus irreducibilis* on our hands. However, there is a neat trigonometric trick which works just in such a case, and we are going to use it.

For starters we 'depress' our cubic equation. We introduce a new unknown u by $t = u - w$, to obtain the equation

$$u^3 \;-\; 3(1+w^2)u \;+\; 2w(1+w^2) \;=\; 0$$

without a quadratic term. Next we introduce another unknown v such that $u = mv$, where the multiplier m is chosen so that the resulting equation will have the form $4v^3 - 3v = c$ (after a suitable rearranging, of course). The desired multiplier is $m = 2\sqrt{1+w^2}$, and the resulting equation is

$$4v^3 - 3v \;=\; -\frac{w}{\sqrt{1+w^2}} \;=\; -\frac{b}{\sqrt{a^2+b^2}} \;=\; -\cos(\alpha) \;=\; \cos(\alpha+\pi)\,.$$

Finally, we set $v = \cos(\varphi)$, get the equation

$$\cos(3\varphi) \;=\; \cos(\alpha+\pi)\,,$$

and its solutions,

$$\varphi_k \;=\; \pm\frac{\alpha+\pi}{3} + k\frac{2\pi}{3}\,, \qquad k \text{ any integer}\,.$$

Substituting these solutions all the way back, we get solutions of the cubic equation we have started with:

$$\begin{aligned} t_1 \;&=\; 2\sqrt{1+w^2}\cos\left(\frac{\alpha}{3}+\pi\right) \;-\; w\,,\\ t_2 \;&=\; 2\sqrt{1+w^2}\cos\left(\frac{\alpha}{3}+\frac{\pi}{3}\right) \;-\; w\,,\\ t_3 \;&=\; 2\sqrt{1+w^2}\cos\left(\frac{\alpha}{3}-\frac{\pi}{3}\right) \;-\; w\,. \end{aligned}$$

We have assigned the indices of t_1, t_2, t_3 correctly, since $t_1 = -2\sqrt{1+w^2}\cos(\alpha/3) - w < -w$ and $t_3 - t_2 = 4\sqrt{1+w^2}\sin(\pi/3)\sin(\alpha/3) > 0$.

Now, let us look at the two diagrams in Figure 5: the diagram of

$$t_3(w) \;:=\; 2\sqrt{1+w^2}\cdot\cos\left(\frac{\arctan(1/w)}{3} - \frac{\pi}{3}\right) \;-\; w$$

for $0 \leqslant w \leqslant 1$, and the diagram of

$$\theta_3(\alpha) \;:=\; \arctan\left(t_3\left(\frac{1}{\tan(\alpha)}\right)\right)$$

for $\pi/4 \leqslant \alpha \leqslant \pi/2$, this one with both α and θ_3 measured in degrees. (The value $t_3(0)$ has no geometric meaning by itself, it is just the limit of values $t_3(w)$ when $w \searrow 0$. Similarly $\theta_3(\pi/2)$ has no geometric meaning, it is just a limit.) And here we make a surprising discovery: *the diagram of* $\theta_3(\alpha)$ *is a straight line!* To be precise, it *looks* like a straight line; if it really is a straight line, then it is the diagram of $\alpha/3 + \pi/6$, because $\theta_3(\pi/4) = \pi/4$ and $\theta_3(\pi/2) = \pi/3$.

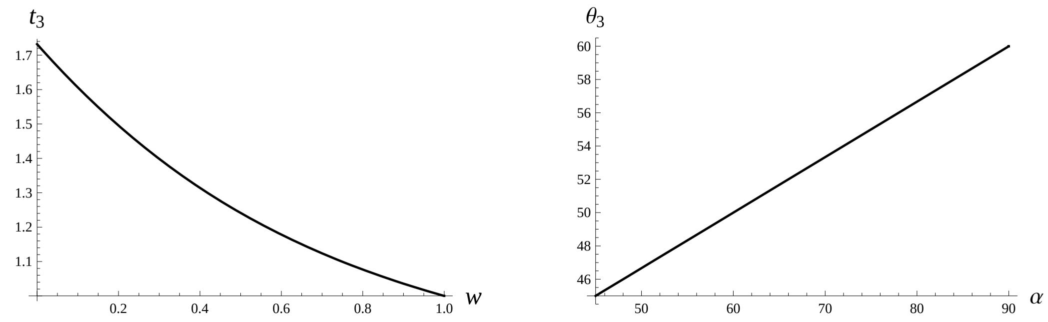

Figure 5: Diagrams of $t_3(w)$ for $0 \leqslant w \leqslant 1$ (left) and of $\theta_3(\alpha)$ for $\pi/4 \leqslant \alpha \leqslant \pi/2$ (right).

If this is true, then $t_3 = \tan(\alpha/3 + \pi/6)$; since $w = \cos(\alpha)/\sin(\alpha)$ and $\sqrt{1+w^2} = 1/\sin(\alpha)$, the following identity must hold:

$$\tan\left(\frac{\alpha}{3} + \frac{\pi}{6}\right) \;=\; \frac{1}{\sin(\alpha)}\left(2\cos\left(\frac{\alpha}{3} - \frac{\pi}{3}\right) \;-\; \cos(\alpha)\right) .$$

Conversely, if the above identity holds, then $\theta_3(\alpha) = \alpha/3 + \pi/6$. So, let's try to prove that we really have an identity here. First notice that $\cos(\alpha/3 - \pi/3) = \sin(\alpha/3 + \pi/6)$; this is encouraging. Put $\varphi = \alpha/3 + \pi/6$; then $\alpha = 3\varphi - \pi/2$, $\cos(\alpha) = \sin(3\varphi)$, $\sin(\alpha) = -\cos(3\varphi)$, the purported identity becomes

$$\frac{\sin(\varphi)}{\cos(\varphi)} \;=\; \frac{2\sin(\varphi) - \sin(3\varphi)}{-\cos(3\varphi)} ,$$

and can be proved using the identities $\cos(3\varphi) = 4\cos^3(\varphi) - 3\cos(\varphi)$, $\cos^2(\varphi) = 1 - \sin^2(\varphi)$, and $\sin(3\varphi) = 3\sin(\varphi) - 4\sin^3(\varphi)$:

$$\begin{aligned} -\sin(\varphi)\cos(3\varphi) \;&=\; \sin(\varphi)\cos(\varphi)\big(3 - 4\cos^2(\varphi)\big) \;=\; \sin(\varphi)\cos(\varphi)\big(4\sin^2(\varphi) - 1\big) , \\ \cos(\varphi)\big(2\sin(\varphi) - \sin(3\varphi)\big) \;&=\; \sin(\varphi)\cos(\varphi)\big(4\sin^2(\varphi) - 1\big) . \end{aligned}$$

It is now easy to verify that $\theta_3(\alpha) = \alpha/3 + \pi/6$ lies within the bounds θ_{lo} and θ_{hi},

$$\begin{aligned} \theta_3(\alpha) - \theta_{\mathrm{lo}} \;&=\; \left(\frac{\alpha}{3} + \frac{\pi}{6}\right) - \frac{\alpha}{2} \;=\; \frac{\pi - \alpha}{6} , \\ \theta_{\mathrm{hi}} - \theta_3(\alpha) \;&=\; \left(\frac{\alpha}{2} + \frac{\pi}{4}\right) \;-\; \left(\frac{\alpha}{3} + \frac{\pi}{6}\right) \;=\; \frac{\alpha + \pi/2}{6} , \end{aligned}$$

and we conclude that $\theta_{\mathrm{opt}} = \theta_3$.

At last we have the solution of the envelope-folding problem: the area of the triangle OYX is minimized at $\theta = \theta_{\mathrm{opt}}$, where

$$\theta_{\mathrm{opt}} \;=\; \theta_{\mathrm{opt}}(\alpha) \;=\; \frac{\alpha}{3} + \frac{\pi}{6} .$$

The linear relation between θ_{opt} and α can be rewritten as a proportional relation between the 'deviations' $\theta_{\mathrm{opt}} - \pi/4$ and $\alpha - \pi/4$ of the angles θ_{opt} and α from one half of the right angle:

$$\theta_{\mathrm{opt}} - \frac{\pi}{4} \;=\; \frac{1}{3}\left(\alpha - \frac{\pi}{4}\right) .$$

How about the minimum area of the triangle OYX? Let us introduce the 'normalized' area p of the triangle OYX, relative to the area of the triangle OAB, as

$$p \;:=\; A_{\triangle\mathrm{OYX}}/A_{\triangle\mathrm{OAB}} \;=\; \frac{1}{4}\,w\,\frac{(1+t^2)^2}{t(t+w)^2} .$$

Taking $t = t_{\text{opt}}$, we get $p = p_{\text{min}}$. We want to know how p_{min} depends on the angle α; this dependence is invariant with respect to complementing the angle α, i.e. $p_{\text{min}}(\pi/2-\alpha) = p_{\text{min}}(\alpha)$. As we have done above, we put $\varphi = \alpha/3 + \pi/6$, and determine how the components of the formula for p_{min} (the above formula for p with $t = t_{\text{opt}}$) depend on φ:

$$\begin{aligned} w &= \frac{\cos(\alpha)}{\sin(\alpha)} = -\frac{\sin(3\varphi)}{\cos(3\varphi)}, \\ 1 + t_{\text{opt}}^2 &= \frac{1}{\cos^2(\varphi)}, \\ t_{\text{opt}} + w &= \frac{\sin(\varphi)}{\cos(\varphi)} - \frac{\sin(3\varphi)}{\cos(3\varphi)} = -\frac{\sin(2\varphi)}{\cos(\varphi)\cos(3\varphi)} = -\frac{2\sin(\varphi)}{\cos(3\varphi)}. \end{aligned}$$

Plugging these into the formula for p_{min} we obtain the desired formula for $p_{\text{min}} = p_{\text{min}}(\alpha)$:

$$p_{\text{min}} = -\frac{1}{16}\frac{\sin(3\varphi)\cos(3\varphi)}{\big(\sin(\varphi)\cos(\varphi)\big)^3} = -\frac{1}{4}\frac{\sin(6\varphi)}{\sin^3(2\varphi)} = \frac{1}{4}\frac{\sin(2\alpha)}{\sin^3\big(\frac{2}{3}\alpha + \frac{1}{3}\pi\big)}.$$

The diagram of the function $p_{\text{min}}(\alpha)$ is shown in Figure 6.

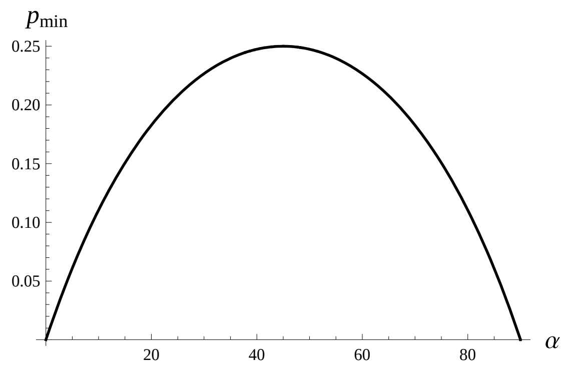

Figure 6: The normalized minimum area $p_{\text{min}}(\alpha)$.

The lemniscate rationale

In the envelope-folding construction, shown in Figure 1, we started with a point P on the hypotenuse AB and constructed the triangle OYX that folds over onto the triangle PYX. Clearly, if we know the triangle OYX, we can reconstruct the point P, which happens to lie on the hypotenuse.

Now we do the following: we vary the triangle OYX, keeping its area constant, and observe the trajectory of the point P. Naturally, we no longer constrain the point P to stay on the hypotenuse, so we'd better think the hypotenuse away; we shall think it back into the fray later on, when the time is ripe. With the hypotenuse out of the way, we have a very large piece of paper possessing a right-angle corner to play with, namely the whole first quadrant of the Oxy coordinate system, as shown in the left panel of Figure 7. To simplify the situation further, we draw the perpendicular UV to the line segment OP through the point P, with the endpoint U on the x-axis and the other endpoint V on the y-axis. The area of the triangle OVU is four times the area of the triangle OYX, thus varying the triangle OVU, while keeping its area constant, produces precisely the same trajectory of P as does varying the triangle OYX, keeping *its* area constant.

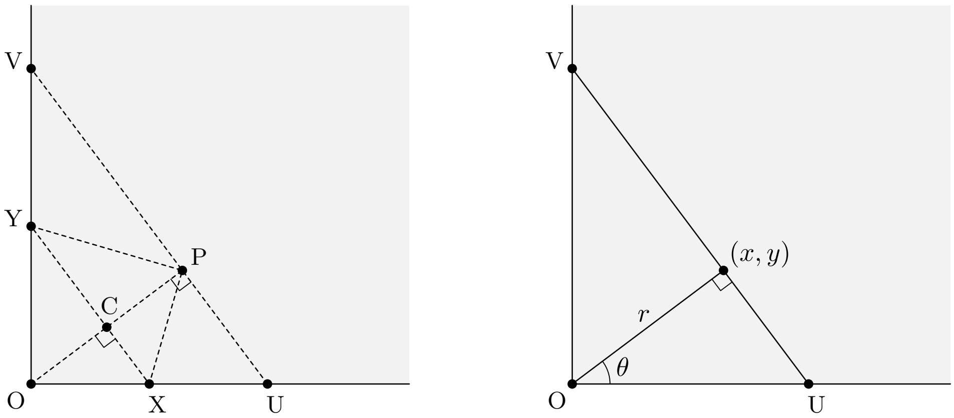

Figure 7: Folding the right angle of the first quadrant.

Let P be the point (x, y) in the interior of the first quadrant, and let (r, θ) be its polar coordinates (Figure 7, right panel). We have $\cos(\theta) = x/r = r/x_{\mathrm{U}}$, hence $x_{\mathrm{U}} = r^2/x$. Similarly, $y_{\mathrm{V}} = r^2/y$, and the area of the triangle OVU is

$$A \;=\; \frac{1}{2} x_{\mathrm{U}} y_{\mathrm{V}} \;=\; \frac{r^4}{2xy} \;=\; \frac{(x^2+y^2)^2}{2xy} \;=\; \frac{r^2}{\sin(2\theta)}\,.$$

If $C > 0$ is a real constant, then the condition $A(r, \theta) = C$ gives us the polar equation

$$r^2 \;=\; C \sin(2\theta)\,, \qquad 0 < \theta < \frac{\pi}{2}$$

of the trajectory ℓ_C of the point P. With the range of θ extended to the closed interval $0 \leqslant \theta \leqslant \pi/2$, the formula for the trajectory gives the origin at the two ends $\theta = 0$ and $\theta = \pi/2$; we add the origin to ℓ_C to obtain a closed curve $\bar{\ell}_C$. This closed curve $\bar{\ell}_C$ is (one half of) the lemniscate of Bernoulli, rotated by $\pi/4$ from its 'standard' position (where it has the polar equation $r^2 = C\cos(2\theta)$). The contour diagram of the function $A(x, y)$ on the interior of the first quadrant (Figure 8) is a family of 'open-ended' lemniscates ℓ_C, where each

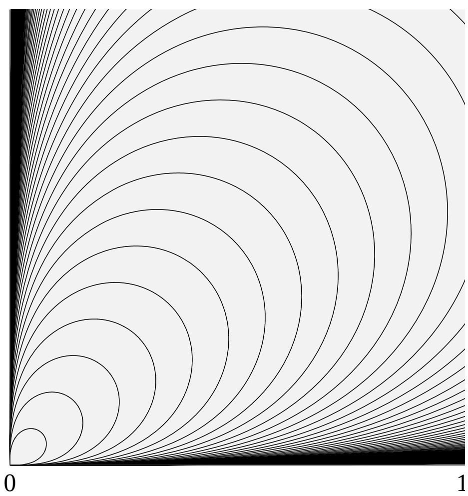

Figure 8: The contour diagram of $A(x, y)$, with the lines at levels $(k \cdot 0.1)^2$, $k = 1, 2, 3, \ldots$.

lemniscate is a scaled image $\sqrt{C}\ell_1$ of the single 'unit lemniscate' ℓ_1. Each point (x_0, y_0) in the interior of the first quadrant belongs to precisely one lemniscate ℓ_{C_0} with $C_0 = A(x_0, y_0)$. The open region $L_C = \big\{(x, y) \;\big|\; x > 0,\ y > 0,\ A(x, y) < C\big\}$ is the union of lemniscates $\ell_{C'}$ with $0 < C' < C$. The boundary of L_C (as a subset of the first quadrant, and also as a subset of the whole plane) is the 'closed' lemniscate $\overline{\ell}_C$, so the closure of L_C is

$$\overline{L}_C \;=\; L_C \cup \overline{\ell}_C \;=\; \big\{(x,y) \;\big|\; x>0,\; y>0,\; A(x,y) \leqslant C\big\} \,\cup\, \big\{(0,0)\big\}\,.$$

We shall call L_C an open — and $\overline{L}_C$ a closed — lemniscate 'petal'. The closed lemniscate petal $\overline{L}_C$ is the union of the closed line segments [O--P] = OP, while its interior L_C is the union of the open line segments (O--P), for all points P of the lemniscate ℓ_C.

We shall calculate the direction angle $\tau = \tau(\theta)$ of (the tangent to) a lemniscate ℓ_C at the point $P = P(\theta)$. For this we need the general formula. Suppose we are given a curve in the Euclidean plane with a polar equation $r = r(\theta)$. Regarding the plane as the plane of complex numbers, the point on the curve is $z(\theta) = r(\theta)e^{i\theta}$, and the direction angle at this point is

$$\tau(\theta) \;=\; \arg\big(z'(\theta)\big) \;=\; \arg\big(\big(r'(\theta)+ir(\theta)\big)e^{i\theta}\big) \;=\; \arg\big(r'(\theta)+ir(\theta)\big) \,+\, \theta\,.$$

For the lemniscate ℓ_C we have

$$\begin{aligned} r \;&=\; \sqrt{C\sin(2\theta)}\,, \\ r' \;&=\; \cos(2\theta)\sqrt{C/\sin(2\theta)}\,, \\ \arg(r'+ir) \;&=\; \arg\big(\cos(2\theta)+i\sin(2\theta)\big) \;=\; 2\theta\,, \\ \tau \;&=\; 2\theta+\theta = 3\theta\,; \end{aligned}$$

see Figure 9. Since $\tau'(\theta) = 3 > 0$, the closed lemniscate petal $\overline{L}_C$ is a strictly convex set.

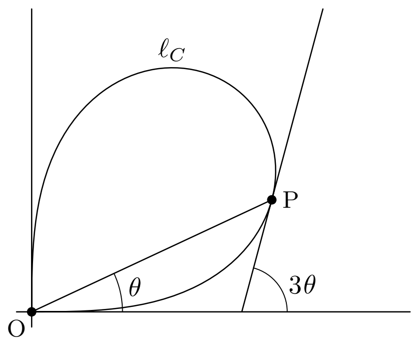

Figure 9: The direction angle of a tangent to a lemniscate.

We are ready for the punch line. We are back with our right triangle OAB. Suppose

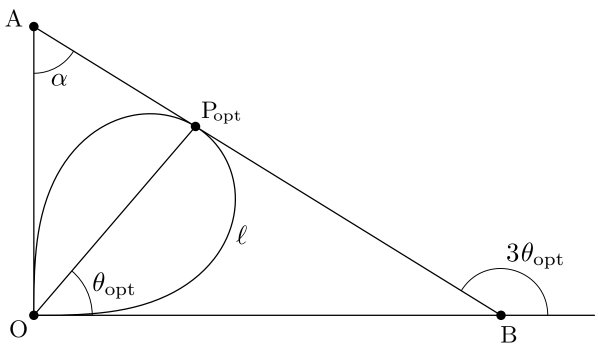

Figure 10: The optimal folding is determined by the lemniscate touching the hypotenuse.

that the minimum folded area occurs at a point $\mathrm{P_{opt}}$ inside the hypotenuse AB, hence in the interior of the first quadrant. Let ℓ be the lemniscate through the point $\mathrm{P_{opt}}$ and let $\overline{L}$ be the corresponding closed lemniscate petal. The hypotenuse intersects $\overline{L}$ in the single point $\mathrm{P_{opt}}$, because $A(\mathrm{P}) > A(\mathrm{P}_{\mathrm{opt}})$ for every point $\mathrm{P} \neq \mathrm{P}_{\mathrm{opt}}$ on the hypotenuse while $A(\mathrm{Q}) \leqslant A(\mathrm{P}_{\mathrm{opt}})$ for every point Q in $\overline{L} \setminus \{\mathrm{O}\}$, so the hypotenuse is the tangent of ℓ at $\mathrm{P}_{\mathrm{opt}}$; the situation is shown in Figure 10. From the figure we see that $3\theta_{\mathrm{opt}} = \alpha + \pi/2$, i.e. that $\theta_{\mathrm{opt}} = \alpha/3 + \pi/6$. We have shown that the minimum folded area is attained at $\theta = \alpha/3 + \pi/6$, *provided* it is attained at some θ in the range $0 < \theta < \pi/2$. To prove that such θ in fact exists, put $\theta_* = \alpha/3 + \pi/6$; we have $\pi/6 < \theta_* < \pi/3$. Let P_* be the point at which the ray from the origin in the direction θ_* intersects the hypotenuse, let ℓ_* be the lemniscate through the point P_* and let $\overline{L}_*$ be the corresponding closed lemniscate petal. Then the hypotenuse is the tangent of ℓ_* at P_* because its direction angle is $3\theta_*$, thus every point $\mathrm{P} \neq \mathrm{P}_*$ on the hypotenuse lies outside $\overline{L}_*$ whence $A(\mathrm{P}) > A(\mathrm{P}_*)$, which proves that the minimum folded area occurs at the point P_* on the hypotenuse.

This result explains the simple form of the formula for θ_{opt} we have found in the preceding section; it also gives an independent solution to the envelope-folding problem.