Approximate construction of a regular nonagon in Albrecht Dürer's *Painter's Manual*: where had it come from?

In *Mathematical Cranks*, in the chapter "Nonagons, regular", Underwood Dudley gives an approximate straightedge and compass construction of a regular nonagon:[1]

> Nonagons follow as corollaries from trisections, being easily made by trisecting 120° angles, and it would be an odd crank indeed who would pass up a famous and general problem for an obscure and particular one.
>
> Nevertheless, nonagoners exist, and Figure 1 is a nonagon construction that was made independently of any trisection. On the circle, mark off |AB|

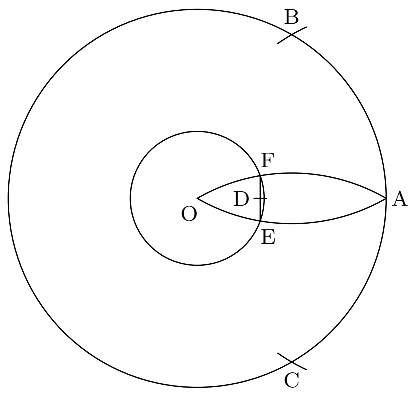

Figure 1: The approximate nonagon construction in *Mathematical Cranks*.

> and |AC| equal to |OC| , and draw arcs from O to A with centers at B and C, both with radius |OA|. Trisect OA at D, draw EF perpendicular to OA, and you have the side of the nonagon inscribed in the circle with radius |OE|. That is, the angle ∡OEF is supposed to be 40°, but it falls short by quite a bit since it measures only 39.6°.

Writing $\alpha = \measuredangle\mathrm{OEF}$, we have

$$\tan\left(\tfrac{1}{2}\alpha\right) \;=\; \frac{1}{2}\left(\sqrt{35} - 3\sqrt{3}\right) \;,$$

and

$$\alpha \;=\; 2\arctan\left(\frac{1}{2}\left(\sqrt{35}-3\sqrt{3}\right)\right) \;=\; 39.594^\circ \;.$$

When we step around the circle with this approximate nonagon's side, the gap that remains after the nine steps is 3.65°, so this is a rather crude approximation.

If this construction was ever advertised by some nonagoner as his (or, most improbably, hers) own, and as an exact construction at that, then there is a very real chance

[1] The quotation is not quite verbatim: the angle EOF in the original is written here as ∡OEF, and the reference to Figure 31 in the book is replaced with the reference to its re-creation in this essay.

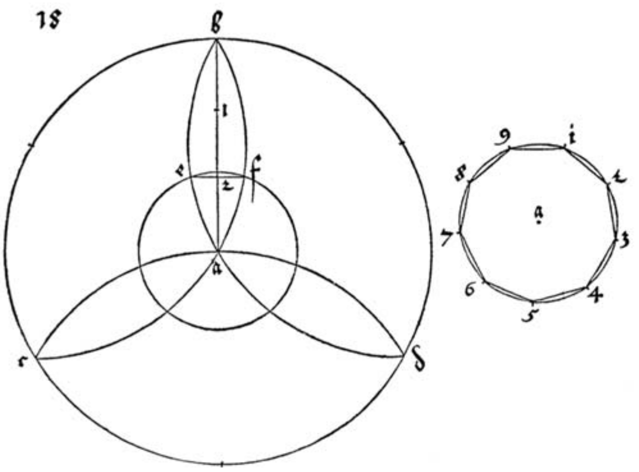

Figure 2: The approximate nonagon construction in *The Painter's Manual*.

that he (or she) had no idea that this construction is straight from Albrecht's Dürer's *The Painter's Manual*; Dürer's drawing of the nonagon's construction is reproduced in Figure 2.[2] Dürer nowhere mentions that the constructed side of a nonagon is just an approximation. That does not make him a nonagoner. The full title of his manual is

THE PAINTER'S MANUAL

A MANUAL OF MEASUREMENT BY COMPASS AND RULER
OF LINES, AREAS, AND SOLIDS
ASSEMBLED BY ALBRECHT DÜRER
FOR THE USE OF ALL LOVERS OF ART
WITH APPROPRIATE ILLUSTRATIONS
ARRANGED TO BE PRINTED IN THE YEAR
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This book is a technical manual, for artists, craftsmen, etc. Dürers emphasis is on easy-to-draw constructions that look good — in fact, some are only good-looking approximations. Many of them were known to craftsmen of the day (techniques passed down through the generations) or appeared in print earlier, but some may have been discovered by Dürer. The approximate construction of a regular nonagon given in *The Painter's Manual* is very probably one of those that were actually just "assembled by Albrecht Dürer for the use of all lovers of art".

The Dürer's construction of the nonagon's vertices, as seen on the right in Figure 2, seems to suggest that the approximately constructed nonagon's side is to be ticked nine times around the circle; but this won't do, the error is too noticeable (Figure 3), even for a circle as small as in Figure 2. I believe that an artisan of those times, when he needed a regular nonagon, completed the construction in Figure 2 as in Figure 4, distributing

[2]The image in the figure is from the web page "Albrecht Dürer's ruler and compass constructions", `http://divisbyzero.com/2011/03/22/albrecht-durers-ruler-and-compass-constructions/`, which is part of *Division by Zero: A blog about math, puzzles, teaching, and academic technology*, posted by Dave Richeson. I have also shamelesly filched a couple of sentences from this page.

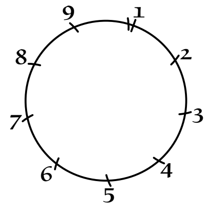

Figure 3: The total error of the Dürer's construction of a regular nonagon.

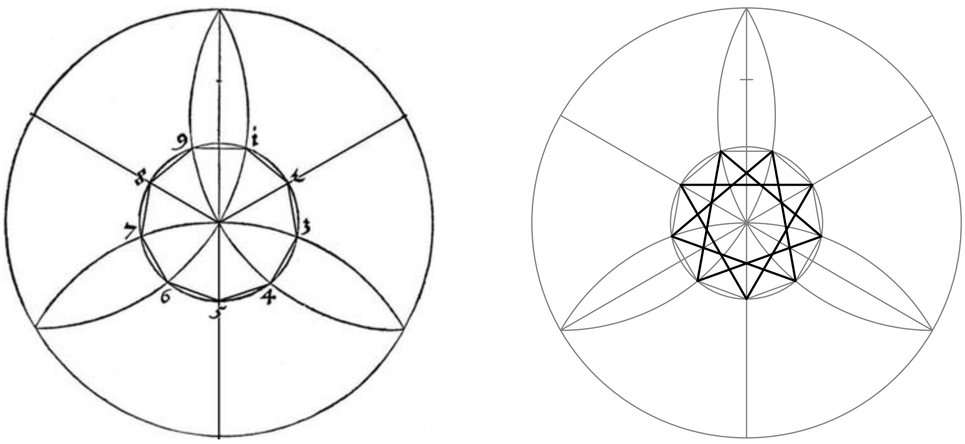

Figure 4: A 'balanced' approximate construction of a regular nonagon.

the overall error among all nine sides. This 'balanced' approximate construction has the nice property that by choosing every third vertex around the nonagon one always obtains the vertices of an exact equilateral triangle. It might well be that those three equilateral triangles inscribed in a nonagon were often the reason why the nonagon was needed — because it was the triad of triangles that was part of some geometric design, not the nonagon itself.

Where had this construction come from? It does not appear to be arrived at by trial and error; the idea that trisecting the radius leads to a close approximation of the nonagon's side must have a geometric background. Musing about this for a while, I have thought of a rather convincing explanation of how the construction might have been discovered. Start with a regular nonagon of side a and radius r (that is, r is the radius of the circumscribed circle), and connect its center to the vertices (Figure 5). Next, on each of nonagon's sides raise a rectangle, away from the nonagon, whose other side is equal to r; the outer vertices of rectangles are vertices of a regular octadecagon with sides of length a. Finally, on each of octadecagon's sides construct an isosceles triangle, again away from the octadecagon, with the other two sides of length r; the apexes of the added triangles are vertices of a regular octadecagon with all sides r. Contemplating the resulting rosette, we notice that all its vertices, except for nine vertices of the outer octadecagon, lie on nine circular arcs that pass through the rosette's center and whose centers are the nine omitted vertices (Figure 6). The nine arcs make for a pleasant composition, a nine-petal flower, which consists of three copies of Dürer's three-petaled construction (Dürer calls those three petals "fish-bladders"), one in the original position, the other two rotated by $40°$ and $80°$. Looking at the superposition, shown in Figure 7,

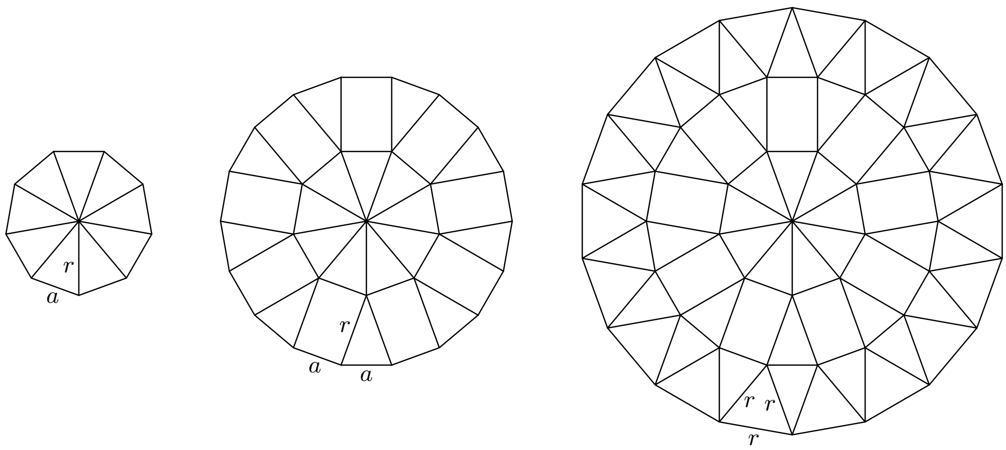

Figure 5: Extending a regular nonagon to a regular octadecagon, then extending once more to a larger regular octadecagon.

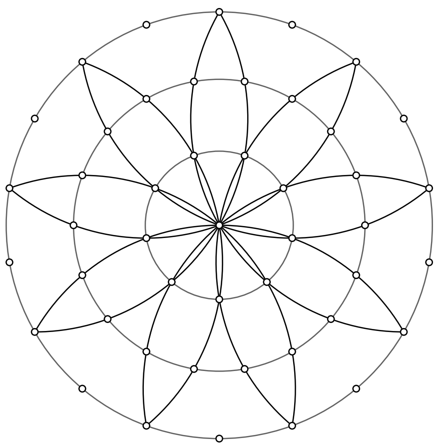

Figure 6: A nine-petal flower.

of the upright petal of the Dürer's construction and the corresponding polygonal petal of the rosette in Figure 5, we see that the two points trisecting the radius lie close to

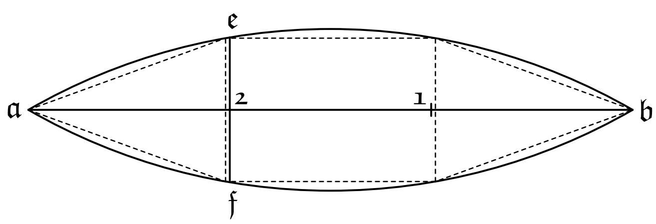

Figure 7: A circular petal vs. the corresponding polygonal petal.

the points at which the perpendiculars to the radius intersect the arcs at the vertices of the regular nonagon and the inner regular octadecagon in the rosette. We also see, at a glance, that the angle $\measuredangle\mathfrak{afe}$ is slightly smaller than $40°$.