Compactness of a Topological Space Via Subbase Covers

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Exercise 1 for §9 of Chapter I in Bourbaki's *General Topology*:

Let X be a topological space and $\mathcal{S}$ a subbase of the topology of X. If every open cover of X by sets belonging to $\mathcal{S}$ contains a finite subcover, then X is compact.

This result is known as the *Alexander subbase theorem*. The exercise provides a hint how to go about proving the theorem. Here we shall approach it from a different direction, using the following special case of Tychonoff's theorem:

The product of any family of finite topological spaces is a compact topological space.

We will actually need only the compactness of the product of finite *discrete* topological spaces. It is easily seen that this even more special case of Tychonoff's theorem implies the special case above: given any finite topological spaces X_ι, $\iota \in I$, the topology $\mathcal{O}$ of the product space $\prod_{\iota\in I} X_\iota$ is cruder than (i.e. is a subset of) the topology of the product of discrete topological spaces on the sets X_ι; since (we are assuming that) the latter topology is compact, the topology $\mathcal{O}$ is also compact.

For a start, observe that the exercise becomes trivial if "subbase" is replaced by "base". Let $\mathcal{B}$ be a base of the topology of X, and suppose that every cover $\mathcal{V} \subseteq \mathcal{B}$ of X has a finite subcover. Given any open cover $\mathcal{U}$ of X, assemble the set $\mathcal{V}$ of all sets $B \in \mathcal{B}$ for which there exists an $U \in \mathcal{U}$ that contains B. Then $\mathcal{V} \subseteq \mathcal{B}$ covers X, it has a finite subcover $\mathcal{W}$, and we obtain a finite subset of $\mathcal{U}$ that covers X by selecting (and collecting) for each set $B \in \mathcal{W}$ a set $U_B \in \mathcal{U}$ that contains B.

Now let $\mathcal{S}$ be a subbase of the topology of X, and suppose that every open cover $\mathcal{V} \subseteq \mathcal{S}$ of X has a finite subcover. The set $\mathcal{B}$ of the intersections of all finite subsets of $\mathcal{S}$ (including the set X, which is by convention the 'intersection' of the empty subset of $\mathcal{S}$) is a base of the topology of X. Suppose that $\mathcal{U} \subseteq \mathcal{B}$ covers X; to prove X compact it suffices to prove that $\mathcal{U}$ has a finite subcover; this is trivially true if $X \in \mathcal{U}$, so suppose that $X \notin \mathcal{U}$. Assume, in order to derive a contradiction, that $\mathcal{U}$ has no finite subcover.

For each $U \in \mathcal{U}$ choose a finite $\mathcal{F}_U \subseteq \mathcal{S}$ so that $\bigcap \mathcal{F}_U = U$; the set $\mathcal{F}_U$ is nonempty because $U \neq X$. Equip each $\mathcal{F}_U$ with the discrete topology, and consider the compact topological space $\Phi := \prod_{U\in\mathcal{U}} \mathcal{F}_U$. First note that every indexed family $F = (F_U \mid U \in \mathcal{U}) \in \Phi$ covers X. Let $\mathcal{V}$ be any finite subset of $\mathcal{U}$, and write $\Phi_\mathcal{V} := \prod_{v\in\mathcal{V}} \mathcal{F}_V$. Since

$$X \;\neq\; \bigcup \mathcal{V} \;=\; \bigcup_{V\in\mathcal{V}} \bigcap \mathcal{F}_V \;=\; \bigcap_{E\in\Phi_\mathcal{V}} \bigcup_{V\in\mathcal{V}} E_V \;,$$

we must have $\bigcup_{V\in\mathcal{V}} E_V \neq X$ for some $E \in \Phi_{\mathcal{V}}$, and it follows that the closed subset[1]

$$\Psi_{\mathcal{V}} := \left\{ F \in \Phi \;\middle|\; \bigcup_{V\in\mathcal{V}} F_V \neq X \right\}$$

of Φ is not empty (because the sets F_U, $U \in \mathcal{U} \smallsetminus \mathcal{V}$, are not empty). If $\mathcal{V}$ and $\mathcal{V}'$ are finite subsets of $\mathcal{U}$ and $\mathcal{V} \subseteq \mathcal{V}'$, then evidently $\Psi_{\mathcal{V}} \supseteq \Psi_{\mathcal{V}'}$, thus the set

$$\left\{ \Psi_{\mathcal{V}} \;\middle|\; \mathcal{V} \text{ is a finite subset of } \mathcal{U} \right\}$$

of closed subsets of Φ has the finite intersection property, and by compactness of Φ it follows that its intersection Ψ_* is not empty. There exists $F \in \Psi_*$, the family $(F_U \mid U \in \mathcal{U})$ of sets in $\mathcal{S}$ covers X while every one of its finite subfamilies fails to cover X, which contradicts our assumption about the subbase $\mathcal{S}$.

In the above proof we have applied the axiom of choice twice: first to choose for each $U \in \mathcal{U}$ a finite subset $\mathcal{F}_U$ of the subbase $\mathcal{S}$ so that $\bigcap \mathcal{F}_U = U$, and then to derive from the existence of $E \in \Phi_{\mathcal{V}}$ such that $\bigcup_{V\in\mathcal{V}} E_V \neq X$ the existence of $F \in \Phi$ (which extends E) such that $\bigcup_{V\in\mathcal{V}} F_V \neq X$. We can eliminate the first application of AC (we are going to show how, in a moment), while it is almost surely impossible to completely get rid of its second application[2] (we won't go into this).

Let the subbase $\mathcal{S}$, the base $\mathcal{B}$, and the open cover $\mathcal{U} \subseteq \mathcal{B}$ (with $X \notin \mathcal{U}$) be as above. Assume that no finite subset of $\mathcal{U}$ covers X. To sidestep the first application of AC, we let $\mathfrak{U}$ be the set of all finite subsets $\mathcal{F}$ of $\mathcal{S}$ such that $\bigcap \mathcal{F} \in \mathcal{U}$; since $X \notin \mathcal{U}$, all $\mathcal{F} \in \mathfrak{U}$ are nonempty. Equip each $\mathcal{F} \in \mathfrak{U}$ with the discrete topology, and let Φ be the product topological space $\prod \mathfrak{U} = \prod_{\mathcal{F}\in\mathfrak{U}} \mathcal{F}$. The points of the compact topological space Φ are indexed families $F = (F_{\mathcal{F}} \mid \mathcal{F} \in \mathfrak{U})$, where $F_{\mathcal{F}} \in \mathcal{F}$ for each $\mathcal{F} \in \mathfrak{U}$. Since every $U \in \mathcal{U}$ is the intersection of some finite subset of $\mathcal{S}$, we have $\bigcup_{\mathcal{F}\in\mathfrak{U}} F_{\mathcal{F}} = X$ for every $F \in \Phi$. Let $\mathfrak{V}$ be any finite subset of $\mathfrak{U}$, and set

$$\Psi_{\mathfrak{V}} := \left\{ F \in \Phi \;\middle|\; \bigcup_{\mathcal{E}\in\mathfrak{V}} F_{\mathcal{E}} \neq X \right\} .$$

As in the above proof we show (using AC) that

$$\left\{ \Psi_{\mathfrak{V}} \;\middle|\; \mathfrak{V} \text{ is a finite subset of } \mathfrak{U} \right\}$$

is a set of nonempty closed subsets of Φ that has the finite intersection property, so by compactness of Φ its intersection Ψ_* is not empty; looking at some point $F \in \Psi_*$ we then derive a contradiction.

[1] The set $\Psi_{\mathcal{V}}$ happens to be also open, which is of no consequence here.

[2] Mark, however, that AC is applied to a family of nonempty *finite* sets.