Characterizations of Compact Metric Spaces

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February 13, 2012
Last modified February 28, 2012

Abstract

These notes gather in one place proofs that several variations on the theme of compactness, which differ for general topological spaces, happen to coincide for metric spaces. We prove that the following topological properties of a metric space are equivalent to each other: compactness (every open cover has a finite subcover); paracompactness (every continuous real-valued function on the space is bounded); sequential compactness (every sequence has a cluster point); limit point compactness (every infinite set has a limit point); countable compactness (every countable open cover has a finite subcover). We also prove the classical characterization of compactness of a metric space in terms of its uniform structure: a metric space is compact if and only if it is complete and totally bounded.

1 Metric spaces: some basic notions and facts

A **metric space** is a structure (M, d), where M is a set and d is a function $M \times M \to \mathbb{R}$ that satisfies the following conditions, for all x, y, $z \in M$:

(1) $d(x, y) \geqslant 0$;

(2) $d(x, y) = 0$ if and only if $x = y$;

(3) $d(x, y) = d(y, x)$;

(4) $d(x, z) \leqslant d(x, y) + d(y, z)$.

The four properties have names: (1) d is non-negative; (2) d is definite; (3) d is symmetric; (4) d satisfies the triangle inequality. Elements of M are usually called **points**. For any two points x and y the number $d(x, y)$ is called the **distance between** x **and** y, while the function d is called the **distance function**, or the **metric**.

Note that the property (1) is redundant, it is a consequence of the other three properties: $2d(x, y) = d(x, y) + d(y, x) \geqslant d(x, x) = 0$.

It is customary, in mathematical discussions, that when we refer to a mathematical structure sitting on a set A, we mention only the underlying set A, provided this cannot cause confusion. Metric spaces are no exception in this regard, so we shall occasionally talk of a metric space M, where M will be in fact just the underlying set of all points of a metric space, whereas we will silently assume that the set M is accompanied by a distance function d.

Let (M, d) be a metric space.

Given $x \in M$ and $r > 0$, the **open ball with center x of radius r** is the set

$$\mathrm{B}(x,r) \ := \ \{ y \in M \mid d(x,y) < r \} \ ,$$

and the **closed ball with center x of radius r** is the set

$$\overline{\mathrm{B}}(x,r) \ := \ \{ y \in M \mid d(x,y) \leqslant r \} \ .$$

We shall also say that an open/closed ball of radius r is an open/closed **r-ball**.

If y is a common point of open balls $\mathrm{B}(x_1, r_1)$ and $\mathrm{B}(x_2, r_2)$, then the intersection of the two balls contains the open ball $\mathrm{B}(y, s)$, where s is the smaller of the numbers $r_1 - d(x_1, y) > 0$ and $r_2 - d(x_2, y) > 0$. The intersection of any two open balls is therefore the union of the open balls contained in it, which means that the collection $\mathcal{B}$ of all open balls of the metric space M is a basis of some topology $\mathcal{O} = \mathcal{O}(d)$ on the set M; the open sets (the members of $\mathcal{O}$) of this topology are the unions of all possible collections of open balls. We shall often refer to the topological space $(M, \mathcal{O})$ as the topological space M (associated with the metric space M).

For any point $x \in M$, the collection of all open balls centered at x is a basis of (open) neighborhoods of the point x in the topological space M; indeed, if V is any neighborhood of x, then V contains an open set U which contains an open ball $\mathrm{B}(y, r)$ which contains the point x, whence V contains the open ball $\mathrm{B}\big(x, r - d(x,y)\big)$ centered at x. Moreover, if R is any set of positive real numbers with $\inf R = 0$, then the collection of open balls $\mathrm{B}(x, r)$ with $r \in R$ is a basis of neighborhoods of the point x; since there are countable sets R with the requisite property, it follows that the topological space M is first countable.

The topological space M is Hausdorff: if x and y are distinct points, then the open balls $\mathrm{B}(x, r)$ and $\mathrm{B}(y, r)$ with $r = \frac{1}{2} d(x, y)$ are disjoint.

The set of all real numbers, equipped with the distance function $|x - y|$, is a metric space. The topology of this metric space is the usual topology of the real line.

Every open ball is an open set by definition. Also, every closed ball $\overline{\mathrm{B}}(x, r)$ is a closed set. To prove the latter, let $y \in M$ be any point outside the closed ball $\overline{\mathrm{B}}(x, r)$; then the open ball $\mathrm{B}(y, s)$, where $s := d(x, y) - r > 0$, is a subset of the complement $M \smallsetminus \overline{\mathrm{B}}(x, r)$ of the closed ball, because every point $z \in \mathrm{B}(y, s)$ satisfies the inequalities $d(x, z) \geqslant d(x, y) - d(y, z) > d(x, y) - s = r$. This proves that $M \smallsetminus \overline{\mathrm{B}}(x, r)$ is open, hence $\overline{\mathrm{B}}(x, r)$ is closed.

A warning is in order here: the closure of the open ball $\mathrm{B}(x, r)$ is always contained in the closed ball $\overline{\mathrm{B}}(x, r)$, but it may not be the whole closed ball; also, the interior of the closed ball $\overline{\mathrm{B}}(x, r)$ always contains the open ball $\mathrm{B}(x, r)$, but may not coincide with it. To see how this can happen, let X be any set, and let $d(x, x) = 0$ for $x \in X$, and $d(x, y) = 1$ for distinct x, $y \in X$; this defines the **standard discrete metric space on X**. Suppose that X has at least two points, pick an $x \in X$ and note that the open ball $\mathrm{B}(x, 1) = \{x\}$ is a closed set, while the closed ball $\overline{\mathrm{B}}(x, 1) = X \neq \mathrm{B}(x, 1)$ is an open set.

Let S be a subset of a metric space M, and let d_S be the restriction of $d\colon M\times M \to \mathbb{R}$ to $S\times S \to \mathbb{R}$. Then (S, d_S) is a metric space, called the **metric subspace induced on S by the metric space** (M, d), or shorter, the **metric subspace S of the metric space** M. If $x \in S$ and $r > 0$, we denote by $\mathrm{B}_S(x, r)$ the open ball centered at x of radius r in the metric subspace S; clearly $\mathrm{B}_S(x, r) = \mathrm{B}(x, r) \cap S$. Denote by $\mathcal{O}_S$ the collection of all open sets of the topology of the metric subspace S. Let $x \in M$ and $r > 0$; then $\mathrm{B}(x, r) \cap S \in \mathcal{O}_S$: if $y \in \mathrm{B}(x, r) \cap S$, then with $s := r - d(x, y)$ we have $\mathrm{B}(y, s) \subseteq \mathrm{B}(x, r)$ and hence $\mathrm{B}_S(y, s) = \mathrm{B}(y, s) \cap S \subseteq \mathrm{B}(x, r) \cap S$. Since the mapping $\mathscr{P}M \to \mathscr{P}S : X \mapsto X \cap S$ preserves arbitrary unions, it follows that $\mathcal{O}_S = \{U \cap S \mid U \in \mathcal{O}\}$, that is, the topological space $(S, \mathcal{O}_S)$ is the subspace of the topological space $(M, \mathcal{O})$.

A subset S of a metric space M is said to be **α-spaced (in M)**, where $\alpha > 0$, if $d(x, y) \geqslant \alpha$ for any two distinct points $x, y \in S$, and it is said to be **spaced out (in M)** if it is α-separated for some $\alpha > 0$. A metric space M is said to be **uniformly discrete** if the set M is spaced out in the metric space M. Any standard discrete metric space is of course uniformly discrete. Recall that a *topological* space is said to be discrete if every one-point set is open, or equivalently, if every subset is open. The topology of a uniformly discrete metric space is discrete. Keep in mind that the converse is not true: a metric space that is not uniformly discrete may have a discrete topology; an example is the metric subspace $\{2^{-n} \mid n \in \mathbb{N}\}$ of $\mathbb{R}$ (with the usual metric).

Let f be a function $A \to B$, where A is either a topological space or a metric space, and the same is true of B. If A is a metric space, we replace it by the associated topological space, and likewise we do with B. Then, if f is a continuous function from the topological space A to the topological space B, we regard it as continuous function from the space A to the space B as they were originally given. For example, the continuity of a function $f\colon M \to M'$ from a metric space (M, d) to a metric space (M', d') is described in epsilon-deltics as follows: f is continuous at $x \in M$ if and only if for every $\varepsilon > 0$ there exists $\delta > 0$ so that $d(x, y) < \delta$ implies $d'\big(f(x), f(y)\big) < \varepsilon$ for every $y \in M$; f is continuous if and only if it is continuous at every $x \in M$.

Any three points x, y, z of a metric space (M, d) satisfy the inequality

$$\big|d(x, y) - d(x, z)\big| \;\leqslant\; d(y, z) \ ,$$

which is a consequence of symmetry and triangle inequality satisfied by the distance function: $d(x, y) \leqslant d(x, z) + d(z, y) = d(x, z) + d(y, z)$, $d(x, z) \leqslant d(x, y) + d(y, z)$. It follows that with a point x fixed, the function $M \to \mathbb{R} : y \mapsto d(x, y)$ is continuous.

Let A be a nonempty subset of a metric space M. For every point $x \in M$ we define the **distance of x from A** as

$$d(x, A) \;:=\; \inf\big\{d(x, a) \;\big|\; a \in A\big\} \ .$$

The distance $d(x, A)$ is a non-negative real number, where $d(x, A) = 0$ if and only if every neighborhood of x contains a point in A, that is, if and only if x belongs to the closure of the set A. Suppose that A is closed and $x \notin A$; then $d(x, A) > 0$, and a little thought shows that $d(x, A)$ is the largest $r > 0$ such that the open ball $\mathrm{B}(x, r)$ is contained in the open set $M \smallsetminus A$. The function $M \to \mathbb{R} : x \mapsto d(x, A)$ is continuous, for any nonempty $A \subseteq M$, because it satisfies the inequality

$$\big|d(x, A) - d(y, A)\big| \;\leqslant\; d(x, y)$$

for all x, $y \in M$. To prove the inequality, let x, $y \in M$, and let a be any point in A. Then $d(x, A) \leqslant d(x, a) \leqslant d(x, y) + d(y, a)$, hence $d(x, A) - d(x, y) \leqslant d(y, a)$, and we conclude that $d(x, A) - d(x, y) \leqslant d(y, A)$ because $a \in A$ is arbitrary. We have the inequality $d(x, A) - d(y, A) \leqslant d(x, y)$, and switching the roles of the points x and y we obtain the inequality $d(y, A) - d(x, A) \leqslant d(x, y)$.

A subset of a metric space M is said to be **compact** if it is a compact subset of the topological space M. That is, a subset K of M is compact if and only if every cover of K by open subsets of M has a finite subcover. A metric space M is said to be compact if the set M of all its points is compact. We know that a subset of a topological space is compact if and only if the topological subspace on the subset is compact. Since a metric subspace has the topology of the topological subspace, it follows that a subset of a metric space is compact if and only if the the metric subspace on the subset is compact. In a metric space as well as in a topological space, compactness of a subset is an intrinsic property of the subspace structure on the subset.

The following properties of compact subsets of a metric space and of compact metric spaces are straight rip-offs of the corresponding properties of compact subsets of a topological space and of compact topological spaces.

A family of subsets of a set A is said to have the **finite intersection property** if the intersection of every finite subfamily is nonempty.[1] As a special case, every nonempty chain of nonempty subsets of a set has the finite intersection property. A metric space M is compact if and only if every family of closed subsets of M that has the finite intersection property has a nonempty intersection; this is essentially just a contrapositive rewording of the definition of compactness of a metric space.

Every closed subset of a compact metric space is compact. Since metric spaces are Hausdorff (when regarded as topological spaces), a compact subset of a metric space is always closed.

If M is a nonempty compact metric space, and if $f\colon M \to \mathbb{R}$ is a continuous function, then there exist points a, $b \in M$ such that $f(a) \leqslant f(x)$ and $f(b) \geqslant f(x)$ for every $x \in M$. Every continuous real-valued function on a compact metric space M is bounded.[2]

2 Lebesgue number and pseudocompactness

Let $\mathcal{U}$ be an open cover of a metric space M. A **Lebesgue number of $\mathcal{U}$** is any positive real number ε which has the property that for every $x \in M$ there exists $U \in \mathcal{U}$ such that $\mathrm{B}(x, \varepsilon) \subseteq U$. An open cover of a metric space may not have a Lebesgue number. Are there metric spaces with the property that every open cover has a Lebesgue number?

Lemma 1 (Lebesgue number lemma)**.** *Every open cover of a compact metric space has a Lebesgue number.*

We shall prove a (seemingly) stronger assertion, with a (seemingly) weaker premise.[3]

[1] In particular, the intersection of the empty subfamily, which is by convention the whole set A, must be nonempty. The empty set does not have a family of subsets possessing the finite intersection property.

[2] This follows from the preceding assertion if $M \neq \varnothing$, and is trivially true if $M = \varnothing$.

[3] The premise of Lemma 2 — that a given metric space is pseudocompact — is in fact equivalent to compactness of the metric space, but we do not know that yet.

A topological space X is said to be **pseudocompact** if every continuous real-valued function on X is bounded. Every compact topological space is pseudocompact.

Lemma 2. *Every open cover of a pseudocompact metric space has a Lebesgue number.*

Proof. Assuming that M is a pseudocompact metric space, let $\mathcal{U}$ be an open cover of M. If $M \in \mathcal{U}$ or $M = \varnothing$, then the conclusion is trivially true, so we assume that $M \notin \mathcal{U}$ and $M \neq \varnothing$, whence $\mathcal{U} \neq \varnothing$ and $M \setminus U \neq \varnothing$ for every $U \in \mathcal{U}$.

First we show that the distance function is bounded, that is, that there exists $D > 0$ so that $d(x, y) \leqslant D$ for all $x, y \in M$. Fix a point $x_0 \in M$. Since the distance $d(x_0, x)$ is a continuous function of $x \in M$, it is bounded, thus there exists $C > 0$ so that $d(x_0, x) \leqslant C$ for all $x \in M$, and hence $d(x, y) \leqslant d(x_0, x) + d(x_0, y) \leqslant 2C$ for all $x, y \in M$.

Now we define on M the real-valued functions f_U, $U \in \mathcal{U}$, and f:

$$f_U(x) := d(x, M \setminus U) , \qquad f(x) := \sup\{f_U(x) \mid U \in \mathcal{U}\} .$$

We have $f_U(x) \leqslant D$ for all $U \in \mathcal{U}$ and all $x \in M$, and hence $f(x) \leqslant D$ for all $x \in M$. Each of the functions f_U satisfies the inequality $\bigl|f_U(x) - f_U(y)\bigr| \leqslant d(x, y)$. Thus for all $x, y \in M$ and any $U \in \mathcal{U}$ we have $f_U(x) \leqslant f_U(y) + d(x, y) \leqslant f(y) + d(x, y)$, whence $f(x) \leqslant f(y) + d(x, y)$; since also $f(y) \leqslant f(x) + d(x, y)$, the function f satisfies the inequality $\bigl|f(x) - f(y)\bigr| \leqslant d(x, y)$ for all x, $y \in M$ and is therefore continuous. Every $x \in M$ belongs to some $U \in \mathcal{U}$, so we have $f(x) \geqslant f_U(x) > 0$. It follows that the function $1/f$ is continuous, thus it is bounded from above, hence the function f is bounded from below by some positive constant α. Choose any positive $\varepsilon < \alpha$ (say $\varepsilon := \frac{1}{2}\alpha$); we claim that ε is a Lebesgue number of $\mathcal{U}$. Let $x \in M$; then there exists $U \in \mathcal{U}$ such that $f_U(x) > \varepsilon$, whence $\mathrm{B}(x, \varepsilon) \subseteq U$. $\square$

Pseudocompact metric spaces are not the only metric spaces whose every open cover has a Lebesgue number; for example, every uniformly discrete metric space has this property, and infinite uniformly discrete metric spaces are not pseudocompact.

Let (M, d) and (M', d') be metric spaces; we shall write open balls in the latter metric space as $\mathrm{B}'(-,-)$.

A function $f \colon M \to M'$ is said to be **uniformly continuous** if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $d(x, y) < \delta$ implies $d'\bigl(f(x), f(y)\bigr) < \varepsilon$ for all $x, y \in M$. The notion of uniform continuity of functions between metric spaces is not topological, meaning that it canot be expressed in terms of topologies of metric spaces. However, it is clear that every uniformly continuous function is continuous.

We have already met examples of uniformly continuous functions: for any nonempty $A \subseteq M$ the function $M \to \mathbb{R} : x \mapsto d(x, A)$ is uniformly continuous; with $A = \{a\}$ we have the function $x \mapsto d(a, x)$ as a special case. Also the function f in the proof of Lemma 2 is uniformly continuous.

Lemma 3. *Let M and M' be metric spaces. If every open cover of M has a Lebesgue number, then every continuous function $M \to M'$ is uniformly continuous.*

Proof. Let $f\colon M \to M'$ be continuous, and let $\varepsilon > 0$. For each $x \in M$ set[4]

$$U_x \;:=\; f^*\Big(\mathrm{B}'\big(f(x), \tfrac{1}{2}\varepsilon\big)\Big) \;.$$

The collection $\mathcal{U}$ of open sets $U_x \subseteq M$, $x \in M$, covers M. There exists a Lebesgue number δ of the cover $\mathcal{U}$. Suppose that $x, y \in M$ and $d(x,y) < \delta$. There exists a point $\hat{x} \in M$ for which $\mathrm{B}(x,\delta) \subseteq U_{\hat{x}}$, which means that $d'\big(f(\hat{x}), f(z)\big) < \frac{1}{2}\varepsilon$ for every $z \in \mathrm{B}(x,\delta)$. In particular, $d'\big(f(\hat{x}), f(x)\big) < \frac{1}{2}\varepsilon$ and $d'\big(f(\hat{x}), f(y)\big) < \frac{1}{2}\varepsilon$, and it follows that $d'\big(f(x), f(y)\big) \leqslant d'\big(f(\hat{x}), f(x)\big) + d'\big(f(\hat{x}), f(y)\big) < \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon = \varepsilon$. $\square$

Since a compact metric space is pseudocompact, we have the following corollary:

Theorem 4 (Heine-Cantor)**.** *If M is a compact metric space and M' is a metric space, then every continuous function $M \to M'$ is uniformly continuous.*

3 Sequential compactness

Let A be any set. A **sequence in** A is a function $x\colon \mathbb{N} \to A$, which is usually written as $(x_n \mid n \in \mathbb{N})$.[5] Given a sequence x in A, a **subsequence of the sequence** x is any sequence in A of the form $x \circ \nu = \big(x_{\nu(m)} \mid m \in \mathbb{N}\big)$, where ν is a strictly increasing function $\mathbb{N} \to \mathbb{N}$. If x is a sequence in A and S is a subset of A, we write[6]

$$x^*(S) \;=\; \{n \in \mathbb{N} \mid x_n \in S\} \;.$$

If x is a sequence of points in a metric space M, and if $y \in M$ and $r > 0$, then we write

$$\mathrm{L}_x(y,r) \;:=\; x^*\big(\mathrm{B}(y,r)\big) \;=\; \big\{n \in \mathbb{N} \;\big|\; d(y, x_n) < r\big\} \;.$$

In a topological space X, a point y is said to be a **cluster point of a sequence** x if for every neighborhood V of the point y the set $x^*(V)$ is infinite; therefore, in a metric space, a point y is a cluster point of a sequence x if and only if for every $\varepsilon > 0$ the set $\mathrm{L}_x(y,\varepsilon)$ is infinite.[7] It is clear that any cluster point of a subsequence of a sequence x is also a cluster point of the sequence x.

Lemma 5. *In a metric space M, a point y is a cluster point of a sequence x if and only if some subsequence of x converges to y.*

[4]Let A and B be sets, and let f be a function $A \to B$. For any sets $X \subseteq A$ and $Y \subseteq B$ we denote by $f_*(X)$ the set $\{f(x) \mid x \in X\}$ (usually written $f(X)$) and by $f^*(Y)$ the set $\{x \in X \mid f(x) \in Y\}$ (usually written $f^{-1}(Y)$), and in this way define functions $f_*\colon \mathscr{P}A \to \mathscr{P}B$ and $f^*\colon \mathscr{P}B \to \mathscr{P}A$. The mapping $A \mapsto \mathscr{P}A$ is the object part of a covariant functor $\mathbf{Set} \to \mathbf{Set}$ which sends each function f to the function f_*, and it is also the object part of a contravariant functor $\mathbf{Set}^{\mathrm{op}} \to \mathbf{Set}$ which sends f to f^*.

[5]In a topological context, “sequence” without a qualification usually means “infinite sequence”. When we occasionally stumble upon some finite sequence, we say that it is finite.

[6]Recall that $x^*(S)$ denotes the inverse image of the set S by the function $x\colon \mathbb{N} \to A$.

[7]Requiring that the set $\mathrm{L}_x(y,r)$ — or the set $x^*(V)$ in the preceding statement — must be infinite is peculiar to sequences; it just so happens that the cofinal subsets of the directed set $\mathbb{N}$ (with the usual ordering) are precisely the infinite subsets of $\mathbb{N}$. In a more general setting, if X is a topological space and D is a directed set, then a point $y \in X$ is said to be a cluster point of a net $x\colon D \to X$ if for every neighborhood V of y the set $x^*(V)$ is cofinal in D.

Proof. Suppose that y is a cluster point of x. We construct a function $\nu\colon \mathbb{N} \to \mathbb{N}$, as follows. We set $\nu(0) := 0$. Let $m > 0$ and suppose that we have already determined $\nu(k)$ for $0 \leqslant k < m$. Since $\mathrm{L}_x(y, 1/k)$ is infinite, it contains natural numbers $n > \nu(m-1)$; we let $\nu(m)$ be the least such n. It is clear that ν is strictly increasing and that $x_{\nu(m)} \to y$ as $m \to \infty$.

Suppose that y is the limit of a subsequence $\big(x_{\nu(m)} \mid m \in \mathbb{N}\big)$ of x. Given any $r > 0$, we have $d\big(y, x_{\nu(m)}\big) < r$ for all but finitely many m, which shows that the set $\mathrm{L}_x(y, r)$ is infinite. $\square$

The set of all cluster points of a sequence x is called the **limit set of the sequence** x and is written as $\operatorname{Lim} x$ or $\operatorname{Lim}_{n\in\mathbb{N}} x_n$.

Lemma 6. *If x is a sequence of points in a metric space M, then*

$$\operatorname{Lim} x \;=\; \bigcap_{n\in\mathbb{N}} \operatorname{cl}\big\{x_k \bigm| k \in \mathbb{N},\, k \geqslant n\big\}\ .$$

Proof. If y is any point in M, then

$$\begin{aligned} y \text{ is a cluster point of } x \;&\iff\; (\forall n)(\forall \varepsilon > 0)(\exists k \geqslant n)\big(d(y, x_k) < \varepsilon\big)\\ &\iff\; (\forall n)\big(y \in \operatorname{cl}\{x_k \mid k \geqslant n\}\big)\\ &\iff\; y \in \textstyle\bigcap_n \operatorname{cl}\{x_k \mid k \geqslant n\}\ . \end{aligned}$$ $\square$

A metric space M is called **sequentially compact** if every sequence of points in M has a cluster point.

Lemma 7. *Every compact metric space is sequentially compact.*

Proof. Let x be a sequence of points in a compact metric space. Then $\operatorname{Lim} x = \bigcap_n \operatorname{cl}\{x_k \mid k \geqslant n\}$ is the intersection of a nonempty chain of nonempty closed sets, hence is not empty. $\square$

Lemma 8. *If M is a nonempty sequentially compact metric space, and f is a continuous real-valued function on M, then there exist points $a, b \in M$ so that $f(a) \leqslant f(x)$ and $f(b) \geqslant f(x)$ for all $x \in M$. Consequently, a sequentially compact metric space is pseudocompact.*

Proof. Let $\beta = \sup_{x\in M} f(x)$, with the supremum taken in the extended real line. There exists a sequence of points (x_n) so that $f(x_n) \to \beta$ as $n \to \infty$. Some subsequence $\big(x_{\nu(m)}\big)$ converges to a point b, and still $f\big(x_{\nu(m)}\big) \to \beta$ as $m \to \infty$. By continuity of f we have also $f\big(x_{\nu(m)}\big) \to f(b)$ as $m \to \infty$, which shows that $\beta = f(b) < \infty$. By the same reasoning applied to $-f$ there is a point a so that $-f(a) \geqslant -f(x)$ for all $x \in M$. $\square$

Let A be a subset of a metric space M, and $r > 0$; we write

$$\mathrm{N}(A, r) \;:=\; \bigcup_{a\in A} \mathrm{B}(a, r) \;=\; \big\{x \in M \bigm| d(x, a) < r \text{ for some } a \in A\big\}\ ,$$

and call the set $\mathrm{N}(A, r)$ (which is an open subset of M) the **r-neighborhood of** A.

A subset S of a metric space M is said to be **totally bounded** if for every $\varepsilon > 0$ there exists a finite subset C of M so that $S \subseteq \mathrm{N}(C, \varepsilon)$; the metric space itself is said to be totally bounded if the set M is totally bounded.

Lemma 9. *A subset S of a metric space M is totally bounded if and only if the metric subspace S of M is totally bounded.*

Proof. Clearly, if the subspace S is totally bounded, then S is totally bounded as a subset of M. Conversely, assume that the subset S is totally bounded in M, and let $\varepsilon > 0$. There exists a finite subset C of M so that $S \subseteq \mathrm{N}\big(C, \frac{1}{2}\varepsilon\big)$. Removing, if necessary, from C all points c for which $\mathrm{B}\big(c, \frac{1}{2}\varepsilon\big)$ is disjoint with S, we can assume that $\mathrm{B}\big(c, \frac{1}{2}\varepsilon\big)$ intersects S for every $c \in C$. For each point $c \in C$ choose a point $c' \in \mathrm{B}\big(c, \frac{1}{2}\varepsilon\big) \cap S$ and let $C' := \{c' \mid c \in C\} \subseteq S$; then $S \subseteq \mathrm{N}(C', \varepsilon) \cap S = \mathrm{N}_S(C', \varepsilon)$. ☐

Lemma 10. *Every pseudocompact metric space is totally bounded.*

Proof. Let M be a metric space. Suppose that for some $\varepsilon > 0$ the space M cannot be covered by finitely many ε-balls. Since M is certainly not empty, we can pick a point $c_0 \in M$. The ball $\mathrm{B}(c_0, \varepsilon)$ is not the whole space, so we can choose a point c_1 outside the ball. The union of the balls $\mathrm{B}(c_0, \varepsilon)$ and $\mathrm{B}(c_1, \varepsilon)$ is still not the whole space, so we can choose a point c_2 outside this union. Continuing in this way, we produce a sequence of points $(c_n \mid n \in \mathbb{N})$ with the property that $d(c_n, c_m) \geqslant \varepsilon$ when $m \neq n$.[8]

For each $n \in \mathbb{N}$ we define a real-valued function f_n on M, setting

$$f_n(x) \;:=\; \max\Big(0,\; n \cdot \Big(1 - \frac{4}{\varepsilon} d(c_n, x)\Big)\Big) \qquad \text{for } x \in M \;.$$

Then $f_n(c_n) = n$, and $f_n(x) = 0$ whenever x is not in the open ball $B_{c_n} := \mathrm{B}\big(c_n, \frac{1}{4}\varepsilon\big)$. All functions f_n are continuous, and their sum $f(x) := \sum_{n \in \mathbb{N}} f_n(x)$ is well defined at every point x because at most one term $f_n(x)$ is not zero. Moreover, for every n the restriction of f to the ball $\mathrm{B}\big(c_n, \frac{3}{4}\varepsilon\big)$ is the same as the restriction of f_n to this ball. Let x be any point in M. The ball $B_x := \mathrm{B}\big(x, \frac{1}{4}\varepsilon\big)$ meets at most one of the balls B_{c_n}; when it meets none, the restriction of f to B_x is identically zero. If B_x meets B_{c_n}, then the restriction of f to B_x coincides with the restriction of f_n to B_x because $B_x \subseteq \mathrm{B}\big(c_n, \frac{3}{4}\varepsilon\big)$, and is therefore continuous (on the subspace B_x, of course). Since the open subsets B_x, $x \in M$, cover the space, and since the restriction of f to each B_x is continuous, f itself is continuous. However, f is not bounded. ☐

Lemma 11. *Every pseudocompact metric space is compact.*

Proof. Let M be a pseudocompact metric space, and let $\mathcal{U}$ be an open cover of M. By Lemma 2, $\mathcal{U}$ has a Lebesgue number δ. By Lemma 10, M is totally bounded, so it can be covered by a collection of open balls $\mathrm{B}(c, \delta)$, $c \in C$, for some finite set C of points. But every one of the balls $\mathrm{B}(c, \delta)$, $c \in C$, is contained in some open set $U_c \in \mathcal{U}$, and we have a finite subcover $\{U_c \mid c \in C\}$. ☐

[8] That is, the points c_n are all distinct and the set $\{c_n \mid n \in \mathbb{N}\}$ is ε-spaced.

We have come a full circle. Let us sum up the results of this section.

Proposition 12. *The following three properties of a metric space M are equivalent to each other:*

- ⋄ *M is compact;*
- ⋄ *M is sequentially compact;*
- ⋄ *M is pseudocompact.*

Proof. Follows from Lemmas 7, 8, and 11. □

4 Limit point compactness

Let X be a topological space and S a subset of X.

A point $y \in X$ is said to be a **limit point** (or **accumulation point**) **of** S if every neighborhood of y contains a point in S that is different from y. The set of all limit points of the set S is called the **derived set of** S and is usually denoted by S'; we shall write it $\operatorname{der} S$.

The subset S is closed in M if and only if $\operatorname{der} S \subseteq S$; moreover, $\operatorname{cl} S = S \cup \operatorname{der} S$. Since $y \in X$ is a limit point of $S \subseteq X$ if and only if every neighborhood of y intersects the set $S \smallsetminus \{y\}$, that is, if and only if y belongs to the closure of the set $S \smallsetminus \{y\}$, we have $\operatorname{der} S = \bigl\{y \in X \bigm| y \in \operatorname{cl}(S \smallsetminus \{y\})\bigr\}$. If y is a limit point of a subset of the set S, then it is clearly also a limit point of the set S; told differently, der is an increasing endofunction of the set $\mathscr{P}X$ partially ordered by inclusion: if $T \subseteq S \subseteq X$, then $\operatorname{der} T \subseteq \operatorname{der} S$.

Lemma 13. *Let M be a metric space, and $S \subseteq M$. A point $y \in M$ is a limit point of S if and only if every neighborhood of y contains infinitely many points of S.*

Proof. If a neighborhood of y contains infinitely many points of S then it certainly contains a point in S that is different from y, which proves the "if" implication.

The "only if" implication. Let y be a limit point of S, and let V be a neighborhood of y. We pick a point $x_0 \in S \cap V$, $x_0 \neq y$. Having a point $x_n \neq y$ for some $n \in \mathbb{N}$, we pick a point $x_{n+1} \in S \cap V \cap \mathrm{B}\bigl(y, d(y, x_n)\bigr)$, $x_{n+1} \neq y$. The infinitely many different points $x_0, x_1, \ldots, x_n, \ldots$ of the set S all lie in the neighborhood V. □

A sequence of points in a set A is a function $\mathbb{N} \to A$, so it makes sense to say that a sequence x is injective if $x_n \neq x_m$ whenever $n \neq m$.

Lemma 14. *If every injective sequence in a metric space M has a cluster point, then every sequence in M has a cluster point (i.e. M is sequentially compact).*

Proof. Assuming the premise, let x be any sequence of points in M. Let X be the set $\{x_n \mid n \in \mathbb{N}\}$, and for each $z \in X$ set $J_z := \{n \in \mathbb{N} \mid x_n = z\}$.

If X is finite, then J_z is infinite for some $z \in X$, and this z is a cluster point of x.

Suppose X is infinite. For each $z \in X$ let $\iota(z)$ be the smallest natural number in J_z, and let $I := \{\iota(z) \mid z \in X\}$. The mapping $\iota\colon X \to \mathbb{N}$ is injective, thus I is an infinite set of natural numbers, and there exists a unique strictly increasing function $\nu\colon \mathbb{N} \to \mathbb{N}$ for which $I = \{\nu(n) \mid n \in \mathbb{N}\}$. The subsequence $\big(x_{\nu(m)}\big)$ of x is injective (it is a bijection of $\mathbb{N}$ onto X), so by assumption it has a cluster point, which is also a cluster point of the sequence x. $\square$

A metric space M is said to be **limit point compact** if every infinite subset of M has a limit point.

Lemma 15. *A metric space M is limit point compact if and only if it is sequentially compact.*

Proof. Suppose M is limit point compact, and let x be an injective sequence of points in M. The infinite set $\{x_n \mid n \in \mathbb{N}\}$ has a limit point, which is a cluster point of the sequence x by Lemma 13.

Suppose M is sequentially compact. Let $S \subseteq M$ be infinite. There is an injective sequence x of points in S; x has a cluster point $y \in M$. For every $\varepsilon > 0$ the set $\mathrm{L}_x(y,\varepsilon)$ is infinite, and so is the set of points $\big\{x_n \bigm| n \in \mathrm{L}_x(y,\varepsilon)\big\} \subseteq S \cap \mathrm{B}(y,\varepsilon)$. This shows that y is a limit point of S. $\square$

5 Countable compactness

A topological space X is said to be **countably compact** if every countable open cover of X has a finite subcover. It is obvious that compactness implies countable compactness.

Lemma 16. *Every countably compact topological space is limit point compact.*

Proof. Let X be a topological space. Suppose that some infinite subset S of X has no limit point in X. Choose a countably infinite subset T of S. The subset T, too, has no limit point, thus every point $x \in X$ has a neighborhood which does not contain any point in $T \setminus \{x\}$. Let $t \in T$; every point of X not in $T \setminus \{t\}$ has a neighborhood disjoint with $T \setminus \{t\}$, so the set $U_t := X \setminus \big(T \setminus \{t\}\big)$ is open. Clearly $\mathcal{U} := \{U_t \mid t \in T\}$ is a countable open cover of X; since $U_t \cap T = \{t\}$, a finite subset of $\mathcal{U}$ covers only finitely many points in T, so it does not cover X, whence X is not countably compact.[9] $\square$

Lemma 17. *A metric space is countably compact if and only if it is compact.*

Proof. If a metric space is countably compact, then it is compact by Lemmas 16 and 15, and by Proposition 12. The opposite implication is obvious. $\square$

[9] Once we have the countable set T lacking a limit point, we construct a countable cover that does not have a finite subcover. However, to choose a countable subset T of an infinite set S we have to invoke some species of the axiom of choice, e.g. the axiom of dependent choice.

6 Complete metric spaces

A sequence x of points in a metric space is said to be a **Cauchy sequence** (or a **fundamental sequence**) if for every $\varepsilon > 0$ there exists a natural number n so that $d(x_j, x_k) < \varepsilon$ for any two natural numbers j, $k \geqslant n$. A metric space M is said to be **complete** if every Cauchy sequence in M converges.

Lemma 18. *The following three properties of a sequence x of points in a metric space M are equivalent to each other:*

(i) *x is a Cauchy sequence;*

(ii) *for every $\varepsilon > 0$ there exists $n \in \mathbb{N}$ so that $d(x_n, x_m) < \varepsilon$ for all $m \in \mathbb{N}$, $m \geqslant n$;*

(iii) *for every $\varepsilon > 0$ there exists $n \in \mathbb{N}$ so that the set $\mathrm{L}_x(x_n, \varepsilon)$ is cofinite in $\mathbb{N}$.*

Proof. The implications (i) $\Longrightarrow$ (ii) and (ii) $\Longrightarrow$ (iii) are obvious.

(iii) $\Longrightarrow$ (i) Let $\varepsilon > 0$. There exists $n \in \mathbb{N}$ so that the set $\mathrm{L}_x\big(x_n, \frac{1}{2}\varepsilon\big)$ is cofinite in $\mathbb{N}$, thus there is $m \in \mathbb{N}$ so that every natural number $k \geqslant m$ belongs to $\mathrm{L}_x\big(x_n, \frac{1}{2}\varepsilon\big)$. Then for any two natural numbers j, $k \geqslant m$ we have $d(x_j, x_k) \leqslant d(x_n, x_j) + d(x_n, x_k) < \varepsilon$. $\square$

Compare the property (iii) to the notion of the limit y of a convergent sequence x: for every $\varepsilon > 0$ the set $\mathrm{L}_x(y, \varepsilon)$ is cofinite in $\mathbb{N}$. Slightly rewording the property (iii), we see that a sequence x is Cauchy if and only if for every $\varepsilon > 0$ the set $\mathrm{L}_x(y, \varepsilon)$ is cofinite in $\mathbb{N}$ for some term y of the sequence x; instead of a 'fixed target' $y \in M$ of a convergent sequence x, there is a 'moving target' $y \in \{x_n \mid n \in \mathbb{N}\}$ of a Cauchy sequence x.

Lemma 19. *Every convergent sequence in a metric space is a Cauchy sequence.*

Proof. Suppose that a sequence x converges to a point y. Let $\varepsilon > 0$. The set $\mathrm{L}_x\big(y, \frac{1}{2}\varepsilon\big)$ is cofinite in $\mathbb{N}$. Pick an $n \in \mathrm{L}_x\big(y, \frac{1}{2}\varepsilon\big)$; then $\mathrm{L}_x(x_n, \varepsilon) \supseteq \mathrm{L}_x\big(y, \frac{1}{2}\varepsilon\big)$ is cofinite in $\mathbb{N}$. $\square$

Lemma 20. *If y is a cluster point of a Cauchy sequence x in a metric space M, then the sequence x converges to the point y.*

Proof. Let $\varepsilon > 0$. For some $n \in \mathbb{N}$ the set $X := \mathrm{L}_x\big(x_n, \frac{1}{3}\varepsilon\big)$ is cofinite in $\mathbb{N}$. Since the set $Y := \mathrm{L}_x\big(y, \frac{1}{3}\varepsilon\big)$ is infinite, there exists $m \in X \cap Y$. For every $k \in X$ we have $d(y, x_k) \leqslant d(y, x_m) + d(x_m, x_n) + d(x_n, x_k) < \varepsilon$, thus $\mathrm{L}_x(y, \varepsilon) \supseteq X$ is cofinite in $\mathbb{N}$. $\square$

We know that the metric subspace on a closed subset of a compact metric space M is always compact, and that every compact subset of a metric space M is closed in M. Therefore, if the metric space M itself is compact, then a metric subspace S of M is compact if and only if S is a closed subset of M. Analogous statements hold for complete metric spaces.

Proposition 21. *A metric subspace on a closed subset S of a complete metric space is complete. If the metric subspace on a subset S of a metric space M is complete, then S is closed in M.*

Proof. Suppose that S is a closed subset of a complete metric space M, and let x be a Cauchy sequence in the subspace S. Then the sequence x is Cauchy in M, it converges in M to some point y that belongs to the closed subset S of M, hence converges in S to the point $y \in S$.

Now suppose that the metric subspace S is complete, and that $S \subseteq M$ is closed. Let y be a point in the closure (taken in the metric space M) of the subset S. There exists a sequence x in S that in M converges to y. The sequence x is a Cauchy sequence in M, so it is also a Cauchy sequence in the complete metric subspace S, and therefore converges to some point $z \in S$. But then x also converges in M to the same point z, hence $y = z \in S$, because a convergent sequence in the metric space M has a unique limit. This proves that S is closed in M. $\square$

Lemma 22. *Every sequentially compact metric space is complete and totally bounded.*

Proof. Let M be a sequentially compact metric space.

Suppose x is a Cauchy sequence in M; then x has a cluster point, which is by Lemma 20 the limit of the sequence x.

By Proposition 12, the metric space M is pseudocompact, hence by Lemma 10 it is totally bounded. $\square$

Lemma 23. *Every complete and totally bounded metric space is sequentially compact.*

Proof. Let M be a complete, totally bounded metric space, and let x be a sequence of points in M. We choose some sequence (ε_n) of positive numbers that converges to 0.

Since M is totally bounded, there exists for each $n \in \mathbb{N}$ a finite subset C_n of M so that M is covered by the ε_n-balls with centers at points in C_n, which implies that $\bigcup_{c \in C_n} \mathrm{L}_x(c, \varepsilon_n) = \mathbb{N}$. For some point $c_0 \in C_0$ the set $\mathrm{L}_x(c_0, \varepsilon_0)$ is infinite. Since

$$\mathrm{L}_x(c_0, \varepsilon_0) \;=\; \bigcup_{c \in C_1} \mathrm{L}_x(c_0, \varepsilon_0) \cap \mathrm{L}_x(c, \varepsilon_1) \;,$$

there is a point $c_1 \in C_1$ such that the set $\mathrm{L}_x(c_0, \varepsilon_0) \cap \mathrm{L}_x(c_1, \varepsilon_1)$ is infinite. Proceeding in this way, we obtain a sequence (c_n) of points, where for each $n \in \mathbb{N}$ we have $c_n \in C_n$ and the set $K_n := \bigcap_{k=0}^{n} \mathrm{L}_x(c_k, \varepsilon_k)$ is infinite; then there exists a strictly increasing function $\nu \colon \mathbb{N} \to \mathbb{N}$ with $\nu(n) \in K_n$ for all $n \in \mathbb{N}$. Since (K_n) is a decreasing sequence of sets, we have $\nu(m) \in K_n$ whenever $m \geqslant n$. It follows that for each $n \in \mathbb{N}$ the set $\mathrm{L}_x\big(x_{\nu(n)}, 2\varepsilon_n\big) \supseteq \mathrm{L}_x(c_n, \varepsilon_n) \supseteq K_n$ contains all $\nu(m)$ with $m \geqslant n$, whence the subsequence $\big(x_{\nu(n)}\big)$ of the sequence x is Cauchy, therefore converges to some point $y \in M$ because M is a complete metric space. $\square$

Recalling that the space $\mathbb{R}^n$, equipped with the Euclidean metric (say), is a complete metric space in which every bounded subset is totally bounded, we obtain the following corollary of Lemmas 22 and 23, and of Proposition 21:

Theorem 24 (Heine-Borel theorem)**.** *A subset of the metric space $\mathbb{R}^n$ is compact if and only if it is closed and bounded.*

7 All flavors of compactness

Now we do the great summing up.

Theorem 25. *The following properties of a metric space M are equivalent to each other:*

(1) *M is compact: every open cover of M has a finite subcover;*

(2) *M is pseudocompact: every continuous real-valued function on M is bounded;*

(3) *M is sequentially compact: every sequence in M has a cluster point;*

(4) *M is limit point compact: every infinite subset of M has a limit point;*

(5) *M is countably compact: every countable open cover of M has a finite subcover;*

(6) *M is complete and totally bounded.*

The first five properties of the metric space (M, d) are in fact properties of the topological space associated with the metric space. The sixth property is different, it is expressible in terms the uniform space $(M, \mathfrak{U})$ associated with the metric space M, where the uniformity $\mathfrak{U} = \mathfrak{U}(d)$ is the filter[10] on the set $M \times M$ generated by the filter base consisting of the r-entourages $V_r := \{(x, y) \mid x, y \in M, d(x, y) < r\}$ for all real $r > 0$.

Bibliography

It all began with the two-page note

Yongheng Zhang, "Compact Metric Spaces",

available at `http://www.math.purdue.edu/~zhang24/compactness.pdf`. It gives a proof that a metric space is compact if and only if it is sequentially compact. I was not entirely satisfied with the proof, so I started looking around.

Among other sources I consulted the Wikipedia articles "Metric space", "Compact space", "Sequentially compact space", "Pseudocompact space", "Limit point compact", "Countably compact space", "Totally bounded space", "Complete metric space", "Uniform space", "Heine-Cantor theorem", and "Heine-Borel theorem". Along the way I stumbled upon several characterizations of compact metric spaces, and decided to gather some of them under the same roof. (There are other characterizations out there, besides the ones presented in these notes, but I decided to stop with the sixth, since I had to stop *somewhere*.)

For additional inspiration I also peeked in the following books:

James Munkres, *Topology*, 2nd Edition. Prentice Hall, 2000.

N. Bourbaki, *General Topology, Chapters 1–4.* Springer-Verlag, 1989.

France Križanič, *Temelji realne matematične analize.* DZS, 1990.

[10] Well, the uniformity $\mathfrak{U}$ of a uniform space $(X, \mathfrak{U})$ is *not* always a filter; however, the only exception is the empty uniform space $(\varnothing, \{\varnothing\})$.