All Triangular Squares

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August 12, 2012

On the first page of Chapter 3 of "Proofs from THE BOOK" we read:

> When is $\binom{n}{k}$ equal to a power m^ℓ? It is easy to see that there are infinitely many solutions for $k = \ell = 2$, that is, of the equation $\binom{n}{2} = m^2$. Indeed, if $\binom{n}{2}$ is a square, then so is $\binom{(2n-1)^2}{2}$. ... ... Beginning with $\binom{9}{2} = 6^2$ we obtain infinitely many solutions — the next one is $\binom{289}{2} = 204^2$.

Not all solutions of $\binom{n}{2} = m^2$ — i.e., the square numbers that are also triangular numbers — are generated in this way. It is not very hard to obtain and describe all solutions; we shall do it here.

Consider the equation

$$\binom{n}{2} \;=\; m^2\;, \qquad\qquad m, n \in \mathbb{N},\, n > 0 \tag{1}$$

(we ignore $\binom{0}{2} = 0^2$). Multiplying (1) by 8 and rearranging, we get the equation

$$(2n-1)^2 - 2(2m)^2 \;=\; 1\;, \qquad\qquad m, n \in \mathbb{N},\, n > 0\;,$$

which is precisely the Pell's equation

$$x^2 - 2y^2 \;=\; 1\;, \qquad\qquad x, y \in \mathbb{N}\;, \tag{2}$$

because all solutions of this equations have x odd and y even. The solutions of (2) are (x_k, y_k), $k \in \mathbb{N}$, where

$$x_k + y_k\sqrt{2} \;=\; \bigl(3 + 2\sqrt{2}\,\bigr)^k\;, \qquad\qquad k \in \mathbb{N} \tag{3}$$

$\bigl($which includes the trivial solution $(x_0, y_0) = (1, 0)\bigr)$. From this we obtain the formula for the k-th solution $(n_k, m_k) = \bigl(\frac{1}{2}(x_k + 1), \frac{1}{2}y_k\bigr)$, $k \in \mathbb{N}$, of (1),

$$n_k \;=\; \frac{\bigl(3 + 2\sqrt{2}\,\bigr)^k + \bigl(3 - 2\sqrt{2}\,\bigr)^k + 2}{4}\;, \tag{4}$$

$$m_k \;=\; \frac{\bigl(3 + 2\sqrt{2}\,\bigr)^k - \bigl(3 - 2\sqrt{2}\,\bigr)^k}{4\sqrt{2}}\;, \tag{5}$$

and for the k-th square number that is also a triangular number,

$$\binom{n_k}{2} = m_k^2 = \frac{\big(17+12\sqrt{2}\big)^k + \big(17-12\sqrt{2}\big)^k - 2}{32} . \tag{6}$$

From (3) we derive the recursion for (n_k, m_k),

$$\begin{aligned} n_{k+1} &= 3n_k + 4m_k - 1 , \\ m_{k+1} &= 2n_k + 3m_k - 1 , \end{aligned} \tag{7}$$

which we can use to produce all solutions starting with the solution $(n_0, m_0) = (1, 0)$. Moreover, looking at the formulas (4), (5), and (6), we notice that n_k and m_k satisfy the recurrence relations

$$\begin{aligned} &n_0 = 1 , \quad n_1 = 2 , \qquad n_k = 6n_{k-1} - n_{k-2} - 2 \quad \text{for } k \geqslant 2 , \\ &m_0 = 0 , \quad m_1 = 1 , \qquad m_k = 6m_{k-1} - m_{k-2} \quad \text{for } k \geqslant 2 , \end{aligned} \tag{8}$$

and that the triangular squares m_k^2 satisfy the recurrence relations

$$m_0^2 = 0 , \quad m_1^2 = 1 , \qquad m_k^2 = 34m_{k-1}^2 - m_{k-2}^2 + 2 \quad \text{for } k \geqslant 2 .$$

Using the formula (4) for n_k, an easy computation shows that $(2n_k - 1)^2 = n_{2k}$. The construction in THE BOOK, quoted above, therefore produces the sequence of numbers n_{2j} for $j = 1, 2, 3, \ldots$.

We can invoke the recursion (7) to demonstrate (without delving into the background where the Pell's equation lurks) that there are infinitely many triangular squares: we prove that

$$\text{if } \binom{n}{2} = m^2 , \qquad \text{then } \binom{3n+4m-1}{2} = (2n+3m-1)^2 ,$$

which follows from the identity

$$\binom{3n+4m-1}{2} - (2n+3m-1)^2 = \binom{n}{2} - m^2 .$$

We know that n_{2k} is a square, namely the square of $2n_k - 1$. Contemplating numbers n_{2k+1} we notice that $n_{2k+1} - 1$ seems always to be a square: $n_1 - 1 = 1^2$, $n_3 - 1 = 7^2$, $n_5 - 1 = 41^2$, $n_7 - 1 = 239^2$, and so on. This is indeed true. The easiest way to see this is to look at the original equation (1) when n is even: setting $n = 2n'$ we get the equation

$$n'(2n' - 1) = m^2 ;$$

since n' and $2n' - 1$ are coprime, both are squares, that is, $n - 1$ and $\frac{1}{2}n$ are squares. (Similarly, when (n, m) is a solution of (1) with n odd, we find that n and $\frac{1}{2}(n-1)$ are squares.) Now, it is clear from the recurrence relations (8) that n_k is always of the same parity as n_{k-2}, for $k \geqslant 2$; since $n_0 = 1$ is odd and $n_1 = 2$ is even, it follows that all n_{2k} are odd and all n_{2k+1} are even. Done.

Therefore, not only are the numbers n_{2k} squares, but so are $\frac{1}{2}(n_{2k} - 1)$, $n_{2k+1} - 1$, and $\frac{1}{2}n_{2k+1}$. Let's figure out what it is that they are the squares of. To achieve this, we push a little further the discussions of the equation (1) for and even n and an odd n.

In the case $n = 2n'$ we have $n' = v^2$ and $2n' - 1 = u^2$, that is, u and v satisfy the Pell's equation

$$u^2 - 2v^2 \;=\; -1 \;, \qquad\qquad u, v \in \mathbb{N}\,,$$

which differs from the equation (2) in that its right hand side is -1, not 1. Solutions of this variant of Pell's equation are (u_k, v_k), $k \in \mathbb{N}$, where

$$u_k + v_k\sqrt{2} \;=\; \big(1+\sqrt{2}\,\big)^{2k+1}\;, \qquad\qquad k \in \mathbb{N}\;.$$

$\big($Note that $\big(1+\sqrt{2}\,\big)^2 = 3 + 2\sqrt{2}$.$\big)$ The formulas for u_k and v_k are

$$u_k \;=\; \frac{\big(1+\sqrt{2}\,\big)^{2k+1} + \big(1-\sqrt{2}\,\big)^{2k+1}}{2}\;,$$

$$v_k \;=\; \frac{\big(1+\sqrt{2}\,\big)^{2k+1} - \big(1-\sqrt{2}\,\big)^{2k+1}}{2\sqrt{2}}\;.$$

A straightforward calculation shows that $u_k^2 + 1 = n_{2k+1}$, and hence $n_{2k+1} = 2v_k^2$.

In the other case $n = 2n' + 1$ we have $n' = y^2$ and $2n' + 1 = x^2$ for some x and y which satisfy the Pell's equation (2) (talk about wheels within wheels…). Formulas for solutions (x_k, y_k), $k \in \mathbb{N}$ are

$$x_k \;=\; \frac{\big(1+\sqrt{2}\,\big)^{2k} + \big(1-\sqrt{2}\,\big)^{2k}}{2}\;,$$

$$y_k \;=\; \frac{\big(1+\sqrt{2}\,\big)^{2k} - \big(1-\sqrt{2}\,\big)^{2k}}{2\sqrt{2}}\;.$$

We have $n_{2k} = x_k^2 = (2n_k - 1)^2$ and $n_{2k} = 2y_k^2 + 1$.

Certainly there are many more wonders one could discover about triangular squares. But let's stop here, shall we? There seems to be a whole science of triangular squares out there; we can be sure that all the results we considered here, and many more besides, were worked out (and worked to death) long ago.

For the formula (6) and references, see Martin Gardner's "New Mathematical Diversions", Revised Edition, The Mathematical Association of America (1995), p. 89.