The proof of Bourbaki–Witt theorem in Lang's Algebra

Here I reproduce, for easier reference, the proof of Bourbaki–Witt theorem as given in Lang's Algebra [1], Appendix 2, pages 881–884. The reproduced proof is *verbatim* as in the book, except that set inclusion is denoted by $\subseteq$ instead of by $\subset$; also the footnotes are not part of the original.

From now on to the end of the proof of Theorem 2.1, we let A be a nonempty partially ordered and strictly inductively ordered set.[1] We recall that **strictly inductively ordered** means that every nonempty totally ordered subset has a least upper bound. We assume given a map $f\colon A \to A$ such that for all $x \in A$ we have $x \leqslant f(x)$. We could call such a map an **increasing**[2] map.

Let $a \in A$. Let B be a subset of A. We shall say that B is **admissible** if:

1. B contains a.

2. We have $f(B) \subseteq B$.

3. Whenever T is a nonempty totally ordered subset of B, the least upper bound of T in A lies in B.

Then B is also strictly inductively ordered, by the induced ordering of A. We shall prove:

Theorem 2.1. (Bourbaki). *Let A be a nonempty partially ordered and strictly inductively ordered set. Let $f\colon A \to A$ be an increasing mapping. Then there exists an element $x_0 \in A$ such that $f(x_0) = x_0$.*

Proof. Suppose that A were totally ordered. By assumption, it would have a least upper bound $b \in A$, and then

$$b \leqslant f(b) \leqslant b\,,$$

so that in this case, our theorem is clear. The whole problem is to reduce the theorem to that case. In other words, what we need to find is a totally ordered admissible subset of A.

If we throw out of A all elements $x \in A$ such that x is not $\geqslant a$, then what remains[3] is obviously an admissible subset. Thus without loss of generality, we may assume that A has a least element a, that is $a \leqslant x$ for all $x \in A$.

Let M be the intersection of all admissible subsets of A. Note that A itself is an admissible subset, and that all admissible subsets of A contain a, so that M is not empty[4]. Furthermore, M is itself an admissible subset of A. To see this, let $x \in M$. Then x is in every admissible subset, so $f(x)$ is in every admissible subset, and hence $f(x) \in M$. Hence $f(M) \subseteq M$. If T is a totally ordered nonempty subset of M, and b is the least upper bound of T in A, then b lies in every admissible subset of A, and hence lies in M. It follows that M is the smallest admissible subset of A, and that any admissible subset of A contained in M is equal to M.

We shall prove that M is totally ordered, and thereby prove Theorem 2.1.

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[1] That is, A is a chain complete partial order.

[2] We could, but won't (except in the Lang's text, of course). *We* call such maps 'ascending'; they are also called, by various authors, 'inflating', 'inflationary', 'expansive', 'progressive', 'explosive'. . .

[3] I.e. the set $\{x \in A \mid a \leqslant x\}$. An intuitionist would vigorously protest that what Lang actually describes here — *if* he describes anything at all — is the set $\{x \in A \mid \neg\neg(a \leqslant x)\}$.

[4] . . . , so M contains a.

Let $c \in M$. We shall say that c is an **extreme point** of M if whenever $x \in M$ and $x < c$, then $f(x) \leqslant c$. For each extreme point $c \in M$ we let

$$M_c = \text{set of } x \in M \quad \text{such that} \quad x \leqslant c \quad \text{or} \quad f(c) \leqslant x\,.$$

Note that M_c is not empty because a is in it.

Lemma 2.2. *We have $M_c = M$ for every extreme point c of M.*

Proof. It will suffice to prove that M_c is an admissible subset. Let $x \in M_c$. If $x < c$ then $f(x) \leqslant c$ so $f(x) \in M_c$. If $x = c$ then $f(x) = f(c)$ is again in M_c.[5] If $f(c) \leqslant x$, then $f(c) \leqslant x \leqslant f(x)$, so once more $f(x) \in M_c$. Thus we have proved that $f(M_c) \subseteq M_c$.

Let T be a totally ordered subset of M_c and let b be the least upper bound of T in M. If all elements $x \in T$ are $\leqslant c$, then $b \leqslant c$ and $b \in M_c$. If some $x \in T$ is such that $f(c) \leqslant x$, then $f(c) \leqslant x \leqslant b$, and so b is in M.[6] This proves our lemma.

Lemma 2.3. *Every element of M is an extreme point.*

Proof. Let E be the set of extreme points of M. Then E is not empty because $a \in E$. It will suffice to prove that E is an admissible subset. We first prove that f maps E into itself. Let $c \in E$. Let $x \in M$ and suppose that $x < f(c)$. We must prove that $f(x) \leqslant f(c)$. By Lemma 2.2, $M = M_c$, and hence we have $x < c$, or $x = c$,[7] or $f(c) \leqslant x$. This last possibility cannot occur because $x < f(c)$. If $x < c$, then

$$f(x) \leqslant c \leqslant f(c)\,.$$

If $x = c$ then $f(x) = f(c)$, and hence $f(E) \subseteq E$.

Next let T be a totally ordered subset of E. Let b be the least upper bound of T in M. We must prove that $b \in E$. Let $x \in M$ and $x < b$. If for all $c \in T$ we have $f(c) \leqslant x$, then $c \leqslant f(c) \leqslant x$ implies that x is an upper bound for T, whence $b \leqslant x$, which is impossible. Since $M_c = M$ for all $c \in E$, we must therefore have $x \leqslant c$ for some $c \in T$.[8] If $x < c$, then $f(x) \leqslant c \leqslant b$, and if $x = c$,[9] then

$$c = x < b\,.$$

Since c is an extreme point and $M_c = M$, we get $f(x) \leqslant b$.[10] This proves that $b \in E$, that E is admissible, and thus proves Lemma 2.3.

We now see trivially that M is totally ordered. For let x, $y \in M$. Then x is an extreme point of M by Lemma 2.3, and $y \in M_x$ so $y \leqslant x$ or

$$x \leqslant f(x) \leqslant y\,,$$

thereby proving that M is totally ordered. As remarked previously, this concludes the proof of Theorem 2.1.

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[5] Here Lang splits the case $x \leqslant c$ into the subcases $x < c$, $x = c$. Since $x < c$ means $(x \leqslant c) \wedge \neg(x = c)$, the two subcases are not exhaustive in intuitionistic logic.

[6] Let $C = \{z \in M \mid z \leqslant c\}$ and $D = \{z \in M \mid f(c) \leqslant z\}$. If we use classical logic, then $T \subseteq C \cup D$ implies $(T \subseteq C) \vee (\exists y\colon y \in T \cap D)$; however, if we use intuitionistic logic, we cannot make this conclusion.

[7] See footnote 5.

[8] See footnote 6. This time $C = \{z \in M \mid f(z) \leqslant x\}$ and $D = \{z \in M \mid x \leqslant z\}$.

[9] See footnote 5.

[10] Since $b \in M = M_c = M_x$, we have $b \leqslant x$ or $f(x) \leqslant b$, where the first case cannot occur because $x < b$.

References

[1] Serge Lang, *Algebra*, Revised Third Edition. Springer-Verlag, New York, 2002.