Bourbaki–Witt theorem for dcpos is constructively equivalent to Bourbaki–Witt theorem for ccpos

A **chain complete partial order**, or shorter, a **ccpo**, is a poset in which every nonempty chain has a join (i.e. a least upper bound).

This is Bourbaki–Witt theorem (we will refer to it as BWT) for ccpos:

BWT for ccpos. *Let P be a ccpo, and let f be an ascending endomap on P. Then for every $a \in P$ there exists a fixed point of f above a.*

It should be proper to refer to this theorem as a "statement", or a "claim". Actually it *is* a theorem in the classical set theory *without* the axiom of choice (AC), but we do not (as of now) know whether it is a theorem in a constructive set theory[1]. This said, we will keep referring to BWT for ccpos, and its cousins, as "theorems", never forgetting that they may not be theorems in a restricted logical context.

A simpler version of BWT for ccpos asserts existence of one fixed point somewhere in a nonempty ccpo; we shall refer to this simpler theorem as BWT1 for ccpos:

BWT1 for ccpos. *Let P be a nonempty ccpo, and let f be an ascending endomap on P. Then f has a fixed point in P.*

It is clear that BWT for ccpos implies BWT1 for ccpos. It is also easy to see that the converse is also true: assume BWT1 for ccpos; let P be any ccpo, and let f be an ascending endomap on P; then for any $a \in P$ the principal filter $A = [a\,..)$ is a nonempty ccpo, mapped into itself by f; by BWT1 for ccpos, the restriction of f to $A \to A$, being an ascending endomap on a nonempty ccpo, has a fixed point, which is the desired fixed point of f above a.

Mark that we use "X is nonempty" to mean "X has at least one element"; in a constructive set theory, a set that is 'nonempty' in this sense is said to be 'inhabited'.

Recall that a **directed set** is a nonempty poset in which any two elements have an upper bound, that a subset of a poset is said to be **directed** if it is directed as a subposet, and that a **directed complete partial order** (a dcpo) is a poset in which every directed subset has a join. We can formulate versions of BWT and BWT1 for dcpos:

BWT for dcpos. *Let P be a dcpo, and let f be an ascending endomap on P. Then for every $a \in P$ there exists a fixed point of f above a.*

BWT1 for dcpos. *Let P be a nonempty dcpo, and let f be an ascending endomap on P. Then f has a fixed point in P.*

It is easy to see, in the same way as above, that BWT for dcpos is equivalent to BWT1 for dcpos. Moreover:

[1] Note that we say "*a* constructive set theory", not "*the* constructive set theory", since there are several constructive set theories, with various levels of constraints prescribing admissible modes of reasoning and legitimate ways of constructing sets. Here we work in a very permissive constructive set theory, one which allows power sets and impredicative constructions.

Proposition 1. *BWT1 for dcpos is constructively equivalent to BWT1 for ccpos.*

Proof. It is clear that BWT1 for ccpos implies BWT1 for dcpos, since every dcpo is a ccpo.

Conversely, assume that BWT1 for dcpos holds, and let P be a nonempty ccpo and f an ascending endomap on P. The set $\mathcal{C}$ of all nonempty chains in P, ordered by inclusion, is nonempty and closed under directed unions, and is therefore a nonempty dcpo. For each chain $C \in \mathcal{C}$, define $F(C) = C \cup \{f(\bigvee C)\} \in \mathcal{C}$. The endomap F on $\mathcal{C}$ is ascending, and by BWT1 for dcpos it has a fixed point B, thus $f(\bigvee B) \in B$ and hence $f(\bigvee B) \leqslant \bigvee B$, which means that $\bigvee B$ is a fixed point of f. $\square$