A ‘classical’ proof that $2^{2^{\mathbb{N}}}$ is homeomorphic to $\mathbb{N}$

We shall give a ‘classical’ proof of the following fact:

> $2^{2^{\mathbb{N}}}$ *is homeomorphic to* $\mathbb{N}$.

In this assertion, $\mathbb{N}$ is the discrete topological space on the set of natural numbers, $2^{\mathbb{N}}$ is the set of all continuous functions $\mathbb{N} \to 2$ (where $2 = \{0, 1\}$ is a discrete two-point space), equipped with the compact-open topology, and $2^{2^{\mathbb{N}}}$ is the set of all continuous functions $2^{\mathbb{N}} \to 2$, also equipped with the compact-open topology. Note that $2^{\mathbb{N}}$ consists of *all* functions $\mathbb{N} \to 2$, because every function from a discrete space is continuous.

Some notation. For any natural number n we denote by $[n]$ the set of all natural numbers strictly less than n.[1] Let m, $n \in \mathbb{N}$. The set $2^{[n]}$ consists of all 01-strings of length n, while $2^{\mathbb{N}}$ is the set of all 01-strings of infinite length. For any $a \in 2^{[n+m]}$ and any $x \in 2^{\mathbb{N}}$, we denote by $a[n]$ the restriction $[n] \hookrightarrow [n+m] \xrightarrow{a} 2$, and by $x[n]$ the restriction $[n] \hookrightarrow \mathbb{N} \xrightarrow{x} 2$; if $A \subseteq 2^{[n+m]}$ and $X \subseteq 2^{\mathbb{N}}$, then we write $A[n] := \{a[n] \mid a \in A\}$ and $X[n] := \{x[n] \mid x \in X\}$. If $a \in 2^{[n]}$, and $x \in 2^{[m]}$ or $x \in 2^{\mathbb{N}}$, we denote by ax the concatenation of the 01-string a followed by the 01-string x; if A is a set of finite 01-strings, and X is a set of finite or infinite 01-strings, then we write $AX = \{ax \mid a \in A,\, x \in X\}$. Finally, we denote by 2^* the set of all finite 01-strings, that is, the (disjoint) union of the sets $2^{[n]}$ for all $n \in \mathbb{N}$.

The proof. The topology of $2^{\mathbb{N}}$ is finite-open, it has a basis $\{a2^{\mathbb{N}} \mid a \in 2^*\}$; all basic open sets are closed. The space $2^{\mathbb{N}}$ is the product of $\mathbb{N}$ compact two-point spaces 2, hence compact.[2] A subset of $2^{\mathbb{N}}$ is clopen (closed and open) if and only if it is a union of finitely many basic open sets: since basic open sets are closed, a union of finitely many of them is always clopen; conversely, a clopen subset, being open, is the union of some set $\mathcal{U}$ of basic open sets, while as a closed subset of a compact space it is compact, whence it is the union of a finite subset of $\mathcal{U}$. Also, we can characterize the clopen subsets of $2^{\mathbb{N}}$ as the subsets of the form $A2^{\mathbb{N}}$, where A is a subset of $2^{[n]}$ for some $n \in \mathbb{N}$. The set of all clopen subsets of $2^{\mathbb{N}}$ is countably infinite.

A function $f\colon 2^{\mathbb{N}} \to 2$ is continuous if and only if $f^{-1}(0)$ and $f^{-1}(1)$ are clopen subsets of $2^{\mathbb{N}}$, that is, if and only if there exist $n \in \mathbb{N}$ and $g\colon 2^{[n]} \to 2$ such that $f(x) = g\big(x[n]\big)$ for all $x \in 2^{\mathbb{N}}$.

Let $f \in 2^{2^{\mathbb{N}}}$, and let n and g be as in the preceding paragraph. Let $a \in 2^{[n]}$ and $j \in 2$; since $a2^{\mathbb{N}}$ is a compact subset of $2^{\mathbb{N}}$ and $\{j\}$ is an open subset of 2, the set $\mathcal{W}(a, j)$, consisting of all $h \in 2^{2^{\mathbb{N}}}$ such that $h\big(a2^{\mathbb{N}}\big) = \{j\}$, is a basic open subset of the topological space $2^{2^{\mathbb{N}}}$. But then $\bigcap \big\{\mathcal{W}\big(a, g(a)\big) \bigm| a \in 2^{[n]}\big\} = \{f\}$ is an open subset of $2^{2^{\mathbb{N}}}$, this for an arbitrary $f \in 2^{2^{\mathbb{N}}}$, and we see that $2^{2^{\mathbb{N}}}$ is a discrete space, of the same cardinality as $\mathbb{N}$. Done.

The points of $2^{2^{\mathbb{N}}}$ are precisely the characteristic functions of the clopen subsets of $2^{\mathbb{N}}$. To exhibit an actual homeomorphism $2^{2^{\mathbb{N}}} \cong \mathbb{N}$, we have to construct some bijective enumeration of the clopen subsets of $2^{\mathbb{N}}$ by all natural numbers. For each clopen subset U of $2^{\mathbb{N}}$ there exists the least $n \in \mathbb{N}$ such that $U = U[n]\,2^{\mathbb{N}}$; we denote this n by $\nu(U)$.[3] For each $n \in \mathbb{N}$ let $\mathcal{U}(n)$ be the set of all clopen subsets U of $2^{\mathbb{N}}$ with $\nu(U) = n$. Then $\mathcal{U}(0) = \{\varnothing, 2^{\mathbb{N}}\}$, and if $n > 0$, then $\mathcal{U}(n)$ consists of all $A2^{\mathbb{N}}$, where $A \subseteq 2^{[n]}$ is not of the form $A'\,2^{[1]}$ for some subset A' of $2^{[n-1]}$. We have $\big|\mathcal{U}(0)\big| = 2$, and $\big|\mathcal{U}(n)\big| = 2^{2^n} - 2^{2^{n-1}}$ for $n > 0$. Now, assign $\varnothing$ to 0 and $2^{\mathbb{N}}$ to 1; for each $n > 0$, lexicographically order $2^{[n]}$, then, using this linear ordering,

[1] Yes, yes, yes, we know that $[n] = n$, at least in some mathematical universes. The point of the notation $[n]$ is that it represents a *set* of natural numbers, not an individual natural number. We won't be quite consistent in the use of this notation, though, because we shall still write 2 instead of $[2]$.

[2] We do not need AC to prove this.

[3] Note that $\nu(2^{\mathbb{N}} \setminus U) = \nu(U)$ for every clopen subset U of $2^{\mathbb{N}}$.

lexicographically order the set of all subsets of $2^{[n]}$, and finally, bijectively enumerate the clopen sets $U \in \mathcal{U}(n)$ by the natural numbers k in the range $2^{2^{n-1}} < k \leqslant 2^{2^n}$, according to the increasing lexicographic order of the sets $U[n] \subseteq 2^{[n]}$. In this way we obtain an effective bijective enumeration of clopen subsets of $2^{\mathbb{N}}$ by all natural numbers.