Multiobjective Optimization in the Presence of Constraints*

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Contents

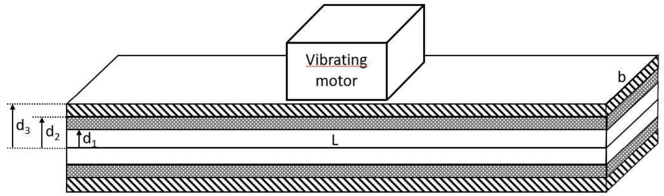
Introduction

Background

- Optimization problems often include both **multiple objectives and constraints**
- **Multiobjective evolutionary algorithms (MOEAs)** – a natural extension of EAs for solving **multiobjective optimization problems (MOPs)**
- Dealing with **constrained multiobjective optimization problems (CMOPs)** long ignored – believed that **constraint handling techniques (CHTs)** for single-objective problems can easily be incorporated into MOEAs
- Recent shift of research focus towards CMOPs

Motivating Example (i)

Vibrating platform (Messac 1996)



- Engineering design problem
- Design variables: d_1, d_2, d_3, b, L
- Task: maximize the fundamental frequency of the platform, minimize its cost

Motivating Example (ii)

Objectives

- f_1 ... fundamental frequency

$$f_1(d_1, d_2, d_3, b, L) = \frac{\pi}{2L^2}\left(\frac{EI}{\mu}\right)^{1/2}$$

$$EI = \frac{2b}{3}\left[E_1 d_1^3 + E_2(d_2^3 - d_1^3) + E_3(d_3^3 - d_2^3)\right]$$

$$\mu = 2b\left[\rho_1 d_1 + \rho_2(d_2 - d_1) + \rho_3(d_3 - d_2)\right]$$

- f_2 ... cost

$$f_2(d_1, d_2, d_3, b) = 2b\left[c_1 d_1 + c_2(d_2 - d_1) + c_3(d_3 - d_2)\right]$$

Motivating Example (iii)

Constraints

- Boundary constraints

$$0.01 \leq d_1 \leq 0.6$$
$$0.01 \leq d_2 \leq 0.6$$
$$0.01 \leq d_3 \leq 0.6$$
$$0.35 \leq b \leq 0.5$$
$$3 \leq L \leq 6$$

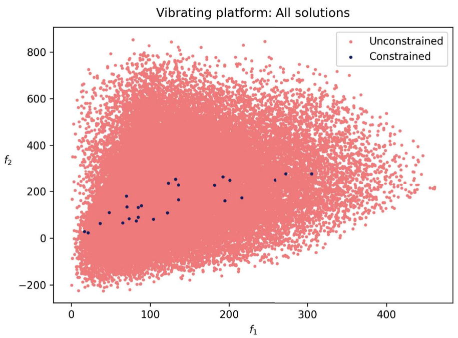
- Inequality constraints

$$0 \leq d_2 - d_1 \leq 0.01$$
$$0 \leq d_3 - d_2 \leq 0.01$$
$$\mu L \leq 2800$$

Motivating Example (iv)

Some problem characteristics

- 5 design variables
- 2 objectives
- 5 constraints
- feasibility ratio* $< 10^{-5}$



*Estimated empirically through solution sampling. Denotes the proportion of feasible solutions among the sampled solutions.

Challenges of Constrained Multiobjective Optimization

- Need to handle both objectives and constraints
- Feasibility ratio can be low
- Objectives and constraints may or may not be correlated
- Feasible region can be disconnected
- etc.

Prerequisites: CMOP Formulation

Constrained multiobjective optimization problem (CMOP):

$$\begin{aligned} \text{minimize} \quad & f_m(x), \quad m = 1, \ldots, M \\ \text{subject to} \quad & g_i(x) \leq 0, \quad i = 1, \ldots, I \\ & h_i(x) = 0, \quad i = I + 1, \ldots, I + J \end{aligned}$$

where

- $x = (x_1, \ldots, x_n)$ ... decision vector
- $S \subseteq \mathbb{R}^n$ ... decision space
- $f_m : S \to \mathbb{R}$ ... objective functions
- $g_i : S \to \mathbb{R}$ ... inequality constraint functions
- $h_i : S \to \mathbb{R}$ ... equality constraint functions

Prerequisites: Constraint Violation

The equality constraints are usually reformulated into inequality constraints:

$$g_i(x) = |h_i(x)| - \epsilon \leq 0, \quad i = I + 1, \ldots, I + J$$

where $\epsilon > 0$ is a user-defined tolerance value (e.g. 10^{-4})

Constraint violation for a single constraint:

$$v_i = \max(g_i(x), 0)$$

Overall constraint violation for all constraints combined:

$$v(x) = \sum_{i=1}^{I+J} v_i$$

Constraint Handling Techniques (CHTs)

CHTs for Single-Objective Optimization

- Penalty functions
- Solution repair
- Separation of objectives and constraints
- Other approaches

Penalty Functions (i)

Idea: transform a constrained problem into an unconstrained one by adding penalty terms to the objective function:

$$f'(x) = f(x) + \sum_{i=1}^{I} p_i \cdot \max(g_i(x), 0) + \sum_{i=I+1}^{I+J} q_i \cdot |h_i(x)|$$

where

- $f'(x)$ ... modified objective function
- p_i ... penalty factors for inequality constraints
- q_i ... penalty factors for equality constraints

Penalty Functions (ii)

Variants

- Death penalty
- Static penalty
- Dynamic penalty
- Adaptive penalty
- Adjustments and modifications of these variants

Penalty Functions (iii)

- Most popular CHT
- Issue: Setting the penalty factors
- Penalties too low: The algorithm spends a lot of time exploring the infeasible region
- Penalties too high: The algorithm may have difficulties detecting the optimum when it is located at the border of the feasible region

Solution Repair

- Idea: Introduce a procedure for converting infeasible solutions to feasible ones
- Repaired solutions can be used for evaluation only, or can replace the original solutions in the population (Lamarckian evolution)
- Problem-dependent, a specific procedure needed for each problem
- Suitable when repair is easy and of low computational cost

Separation of Objectives and Constraints

- In contrast to penalty functions, these techniques handle objectives and constraints separately
- Examples:
    - Superiority of feasible solutions: Always assign a higher fitness to feasible solutions than to infeasible ones
    - Multiobjective optimization approach: $K+1$ objectives where K is the number of constraints
    - Coevolution: evolve two interacting populations

Other Approaches

- Special representations and operators
- Hybrid techniques
- Ensembles of CHTs
- Landscape-aware constraint handling: Using the concept of violation landscape (Malan 2018; Malan and Moser 2019)

CHTs for Multiobjective Optimization

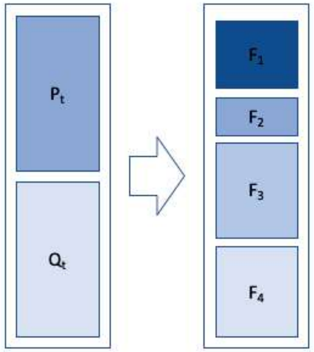
- CHTs incorporated in Nondominated sorting genetic algorithm II (NSGA-II)
- CHTs incorporated in Multiobjective evolutionary algorithm based on decomposition (MOEA/D)
- Advanced techniques

CHTs incorporated in NSGA-II

- Constrained dominance principle (CDP)
- Stochastic ranking (SR)
- Penalty function

NSGA-II

- **The most frequently used algorithm in constrained multiobjective optimization**
- CHT usually incorporated within the sorting procedure



NSGA-II: CDP (Deb, Pratap, Agarwal, et al. 2002)

A solution *x* is said to **constrained-dominate** a solution *y*, if any of the following conditions is true:

- Solution *x* is feasible and solution *y* is not
- Solutions *x* and *y* are both infeasible, but solution *x* has a smaller overall constraint violation than *y*
- Solutions *x* and *y* are feasible and solution *x* dominates solution *y*

The most commonly used CHT in constrained multiobjective optimization (Z. Ma et al. 2019)

NSGA-II: SR (Geng et al. 2006)

Stochastic ranking selection:

- Feasible solutions are compared based on the dominance relation
- Infeasible solutions are compared either based on on the overall constraint violation or dominance relation
- **The comparison criterion is randomly selected**

IS-MOEA:

- Based on NSGA-II and stochastic ranking selection
- Uses the **infeasible elitists preservation**

NSGA-II: Penalty Function (Woldesenbet et al. 2009)

Transform the objective functions into:

$$f_i'(x) = \begin{cases} f_i(x) & \text{if } x \text{ is feasible} \\ v(x) & \text{if } x \text{ is infeasible and } \rho_F(P) = 0 \\ p_i(x) + d_i(x) & \text{if } x \text{ is infeasible and } \rho_F(P) \neq 0 \end{cases}$$

where

$$p_i(x) = (1 - \rho_F(P))v(x) + \rho_F(P)f_i(x)$$

and

$$d_i(x) = \sqrt{f_i(x)^2 + v(x)^2}$$

CHTs incorporated in MOEA/D

- CDP
- SR
- ϵ-constraint (Epsilon)
- Improved ϵ-constraint (IEpsilon)

MOEA/D

The original problem is decomposed into multiple subproblems

The **Tchebycheff aggregation function** is the most widely used decomposition approach in constrained multiobjective optimization

A subproblem is defined as follows:

$$\text{minimize } g(x \mid \lambda, z^*) = \max_{1 \leq i \leq m} \{\lambda_i \, |f_i(x) - z_i^*|\}$$

where z^* is an approximation for the ideal point and λ a weight vector

Idea: The aggregation function can be seen as a fitness of the subproblem $\rightarrow$ Easy to incorporate CHTs for single-objective optimization

MOEA/D-DE (i)

MOEA/D-DE:

- Employs **differential evolution** (DE) operator for generating new solutions
- Limits the maximal number of solutions replaced by a better child solution, n_r

The most interesting part of MOEA/D-DE is the **update phase**

Algorithm 1: Update neighboring solutions N with x

$c \leftarrow 0$;
while $c < n_r$ *and* $N \neq \emptyset$ **do**
  randomly pick $y \in N$;
  if $g(x) < g(y)$ **then**
    $y \leftarrow x$, $c \leftarrow c + 1$;
  end
  $N \leftarrow N - \{y\}$;
end

MOEA/D-DE (ii)

Algorithm 2: Update neighboring solutions N with x

$c \leftarrow 0$;
while $c < n_r$ *and* $N \neq \emptyset$ **do**
  randomly pick $y \in N$;
  if $g(x) < g(y)$ **then**
    $y \leftarrow x$, $c \leftarrow c + 1$;
  end
  $N \leftarrow N - \{y\}$;
end

Algorithm 3: Update neighboring solutions N with x

$c \leftarrow 0$;
while $c < n_r$ *and* $N \neq \emptyset$ **do**
  randomly pick $y \in N$;
  if $x \preceq y$ **then**
    $y \leftarrow x$, $c \leftarrow c + 1$;
  end
  $N \leftarrow N - \{y\}$;
end

MOEA/D: CDP and SR (Jan et al. 2013)

MOEA/D-CDP:

$$x \preceq_{\mathrm{CDP}} y \Leftrightarrow \begin{cases} g(x) < g(y) & \text{if } v(x) = v(y) = 0 \\ v(x) < v(y) & \text{otherwise} \end{cases}$$

MOEA/D-SR:

$$x \preceq_{\mathrm{SR}} y \Leftrightarrow \begin{cases} g(x) < g(y) & \text{if } v(x) = v(y) = 0 \text{ or rand} < p \\ v(x) < v(y) & \text{otherwise} \end{cases}$$

MOEA/D: ϵ-Constraint Technique (Asafuddoula et al. 2012)

MOEA/D-Epsilon:

$$x \preceq_{\epsilon} y \Leftrightarrow \begin{cases} g(x) < g(y) & \text{if } v(x) = v(y) \text{ or } (v(x) \leq \epsilon \text{ and } v(y) \leq \epsilon) \\ v(x) < v(y) & \text{otherwise} \end{cases}$$

The ϵ value is updated in each generation:

$$\epsilon = \overline{v} \cdot \rho_F(P)$$

where

$$\overline{v} = \frac{1}{|P|} \sum_{x \in P} v(x)$$

MOEA/D: Improved ϵ-Constraint Technique (Fan, W. Li, Cai, Huang, et al. 2019)

MOEA/D-IEpsilon:

The ϵ value is updated in each generation:

$$\epsilon(t) = \begin{cases} v(x^\theta) & \text{if } t = 0 \\ (1-\tau)\epsilon(t-1) & \text{if } \rho_F(P) < \alpha \text{ and } t < T_c \\ (1+\tau)v_{\max} & \text{if } \rho_F(P) \geq \alpha \text{ and } t < T_c \\ 0 & \text{if } t \geq T_c \end{cases}$$

where τ, α, T_c are user-defined parameters and $v(x^\theta)$ is the overall constraint violation of the top θ-th individual in the initial population

Advanced Techniques

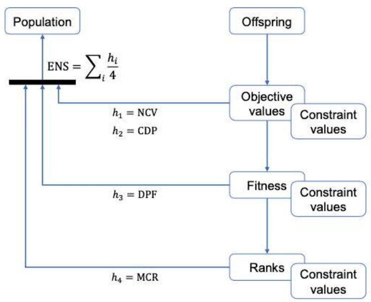
- Ensembles
- Multiple phase techniques
- Multiple population techniques
- Hybrids
- Coevolution

Advanced Techniques: Ensemble of CHTs (Vodopija et al. 2019)

Idea: Combine multiple CHTs into an ensemble

NSGA-II-ENS:

- h_1: Nondominated sorting
- h_2: Constrained-domination principle
- h_3: Dynamic penalty function
- h_4: Multiple constraint ranking



Advanced Techniques: Two-Phase Framework (Liu et al. 2019)

Two-phase framework:

1. **First phase:** Solve a constrained single-objective problem

$$\text{minimize} \quad f'(x) = \sum_{i=1}^{M} f_i(x)$$
$$\text{subject to} \quad g_i(x) \leq 0, \quad i = 1, \ldots, I + J$$

2. **Second phase:** Apply constrained multiobjective optimization on the original problem **starting with solutions obtained in the first phase**

ToP:

- Differential evolution in the first phase
- NSGA-II (CDP) or IDEA in the second phase

Advanced Techniques: Push and Pull Search (Fan, W. Li, Cai, H. Li, et al. 2019b)

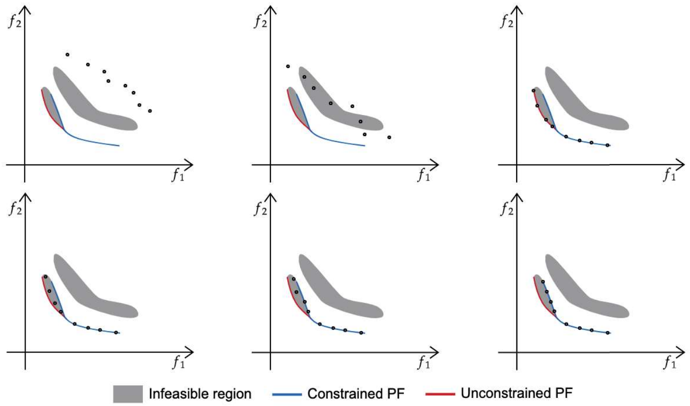
Search is divided into **two stages**:

1. **Push** ignores constraints
2. **Pull** handles infeasible solutions

PPS-MOEA/D:

- Push stage: MOEA/D-DE
- Pull stage: MOEA/D-IEpsilon
- **Parameters for the pull stage assessed in the push stage**

Advanced Techniques: Push and Pull Search (Fan, W. Li, Cai, H. Li, et al. 2019b)



Advanced Techniques: Two-Archive Evolutionary Algorithm (K. Li et al. 2019)

Two complementary archives:

- **Convergence archive:** Maintain the convergence and feasibility of the evolution process
- **Diversity archive:** Maintain the diversity of the evolution process
- A restricted mating mechanism combines parents from the two archives

C-TAEA:

- Based on MOEA/D and M2M framework (decomposition of the original multiobjective optimization problem into multiple simpler subproblems)

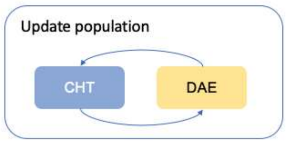
Advanced Techniques: Detect-and-Escape Strategy (Zhu et al. 2020)

If a “suboptimal” area is **detected**, **escape**:

1. Feasible subregion → search areas which **dominate the current one**
2. Infeasible area → **improve diversity**

MOEA/D-DAE:

- Based on MOEA/D and ϵ-constraint CHT
- Only one detect-and-escape cycle is allowed



Advanced Techniques: Recently Proposed

- **CCMO**: Coevolutionary constrained multiobjective optimization (Tian, T. Zhang, et al. 2021)
- **CMOEA-MS**: Two-stage constrained multiobjective optimization (Tian, Y. Zhang, et al. 2021)
- **MSCMO**: Multi-stage evolutionary algorithm for constrained multiobjective optimization (H. Ma et al. 2021)
- **POCEA**: Paired offspring generation-based EA for constrained multiobjective optimization (He et al. 2021)

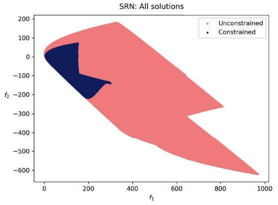
Test Problems

Overview

- Artificial test problems
- Artificial test suites
- Real-world test problems based on mathematical models
- Real-world test problems based on simulation

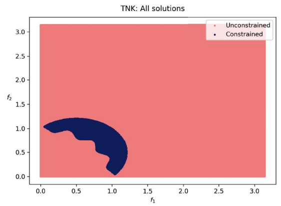
Artificial Test Problems: SRN (Srinivas et al. 1995)

$$
\begin{aligned}
\text{min. } & f_1(x) = 2 + (x_1 - 2)^2 + (x_2 - 1)^2 \\
\text{min. } & f_2(x) = 9x_1 - (x_2 - 1)^2 \\
\text{s.t. } & g_1(x) = x_1^2 + x_2^2 - 225 \leq 0 \\
& g_2(x) = x_1 - 3x_2 + 10 \leq 0 \\
& x_1, x_2 \in [-20, 20]
\end{aligned}
$$



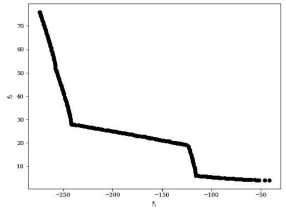
Artificial Test Problems: TNK (Tanaka et al. 1995)

$$
\begin{aligned}
\text{min. } & f_1(x) = x_1 \\
\text{min. } & f_2(x) = x_2 \\
\text{s.t. } & g_1(x) = x_1^2 + x_2^2 - 1 - 0.1\cos(16\tan^{-1}\frac{x}{y}) \geq 0 \\
& g_2(x) = (x_1 - 0.5)^2 + (x_2 - 0.5)^2 - 0.5 \leq 0 \\
& x_1, x_2 \in [0, \pi]
\end{aligned}
$$



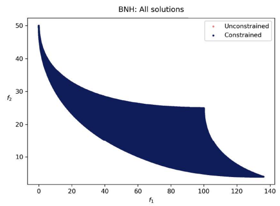
Artificial Test Problems: OSY (Osyczka et al. 1995)

$$
\begin{aligned}
\text{max. } & f_1(x) = 25(x_1 - 2)^2 + (x_2 - 2)^2 + \\
& (x_3 - 1)^2 + (x_4 - 4)^2 + (x_5 - 1)^2 \\
\text{min. } & f_2(x) = x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2 \\
\text{s.t. } & g_1(x) = x_1 + x_2 - 2 \geq 0 \\
& g_2(x) = 6 - x_1 - x_2 \geq 0 \\
& g_3(x) = 2 - x_2 + x_1 \geq 0 \\
& g_4(x) = 2 - x_1 - 3x_2 \geq 0 \\
& g_5(x) = 4 - (x_3 - 3)^2 - x_4 \geq 0 \\
& g_6(x) = (x_5 - 3)^2 + x_6 - 4 \geq 0 \\
& x_1, x_2, x_6 \in [0, 10], x_3, x_5 \in [1, 5], x_4 \in [0, 6]
\end{aligned}
$$



Artificial Test Problems: BNH (Binh et al. 1997)

$$
\begin{aligned}
\text{min. } & f_1(x) = 4(x_1^2 + x_2^2) \\
\text{min. } & f_2(x) = (x_1 - 5)^2 + (x_2 - 5)^2 \\
\text{s.t. } & g_1(x) = (x_1 - 5)^2 + x_2^2 - 25 \leq 0 \\
& g_2(x) = (x_1 - 8)^2 - (x_2 + 3)^2 - 7.7 \geq 0 \\
& x_1 \in [0, 5], x_2 \in [0, 3]
\end{aligned}
$$



Artificial Test Problems: Issues with SRN, TNK, OSY, BNH

Issues:

- Low dimensionality
- Not hard to solve
- Complexity/difficulty not tunable

→ Further proposals: Frameworks for constructing harder tunable problems

Artificial Test Suites: CTP (Deb, Pratap, and Meyarivan 2001)

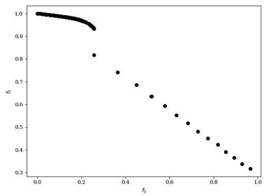
Constrained test problems (CTPs)

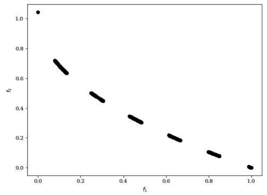
Scalable number of decision variables and **tunable constraint difficulties**

Two kinds of difficulty:

- Difficulty in the vicinity of PF
- Difficulty in the entire search space

8 bi-objective CMOPs including 1–2 constraints

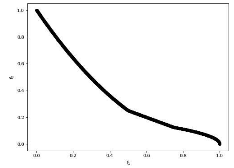


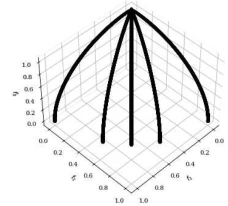


Artificial Test Suites: CF (Q. Zhang et al. 2008)

Constrained multiobjective test problems from the CEC 2009 Special Session and Competition (CFs)

10 problems with 2 or 3 objectives and 1 or 2 constraints





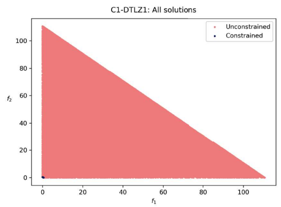
Artificial Test Suites: C-DTLZ (Jain et al. 2014)

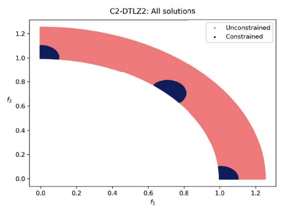
Constrained DTLZ problems (C-DTLZs)

Three types of CMOPs:

- C1: unconstrained PF still optimal, barrier in approaching PF
- C2: only parts of unconstrained PF feasible
- C3: unconstrained PF no longer optimal

6 scalable CMOPs in the number of objectives and constraints





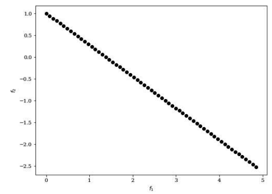
Artificial Test Suites: NCTP (J. Li et al. 2016)

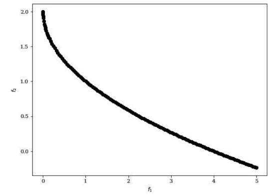
New constrained test problems (NCTPs)

An extension of CTPs:

- Difficulty of convergence is increased
- Infeasible region is increased by an additional constraint

18 bi-objective CMOPs with 1 or 2 constraints





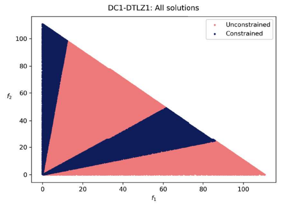
Artificial Test Suites: DC-DTLZ (K. Li et al. 2019)

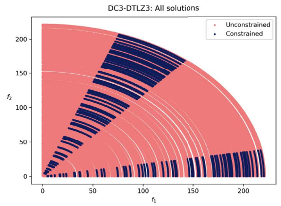
Constrained DTLZ problems where constraints act in the decision space (DC-DTLZs)

Three types of constraints:

- DC1: several infeasible segments
- DC2: unconstrained PF still optimal, barrier in approaching PF
- DC3: decision space consists of several feasible regions

6 scalable CMOPs in the number of objectives and constraints





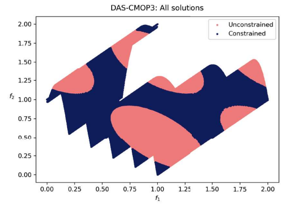
Artificial Test Suites: DAS-CMOP (Fan, W. Li, Cai, H. Li, et al. 2019a)

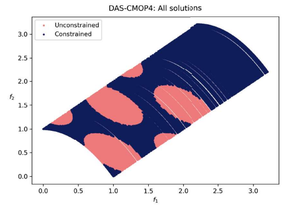
Difficulty-adjustable and scalable CMOPs (DAS-CMOPs)

Test problem kit considering basic difficulty types:

- T1: diversity hardness
- T2: feasibility hardness
- T3: convergence hardness

9 CMOPs of increasing hardness, scalable in the number of objectives

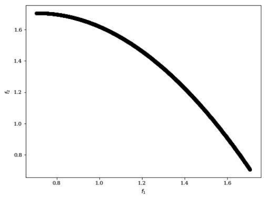


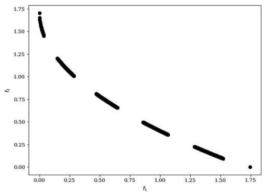


Artificial Test Suites: LIR-CMOP (Fan, W. Li, Cai, Huang, et al. 2019)

Large infeasible region CMOPs (LIR-CMOPs)

14 CMOPs with 2 or 3 objectives and 2 or 3 constraints





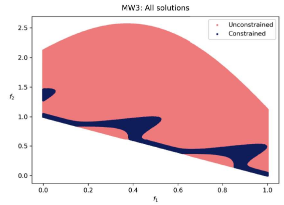
Artificial Test Suites: MW (Z. Ma et al. 2019)

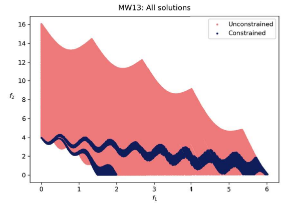
Ma and Wang problems (MWs)

Four types of CMOPs:

- Type I: unconstrained PF remains feasible
- Type II: constrained PF is a part of the unconstrained PF
- Type III: constrained PF consists of a part of the unconstrained PF and part of a boundary
- Type IV: unconstrained PF no longer optimal

11 bi-objective CMOPs and 3 scalable in the number of objective with 1–4 constraints





Artificial Test Suites: Others

- **DOC**: Constrained multiobjective optimization problems with constraints in the decision and objective space (Liu et al. 2019)
- **Eq-DTLZ** and **Eq-IDTLZ**: Benchmark for equality constrained multiobjective optimization (Cuate et al. 2020)
- **CLSMOP**: Constrained large-scale mutliobjective optimization problems (He et al. 2021)

Real-World Test Problems Based on Mathematical Models

Real-world constrained multiobjective optimization problems (RCMs) from CEC 2021 Special Session and Competition and GECCO 2021 Competition[1]

A collection of real-world test problems based on mathematical models:

- Mechanical design problems
- Chemical engineering optimization problems
- Process synthesis optimization problems
- Power systems optimization problems

50 problems with 2–34 variables, 2–5 objectives, and 1–29 constraints

[1] https://www3.ntu.edu.sg/home/epnsugan/index_files/CEC2021/CEC2021-1.htm

Real-World Test Problems Based on Simulations (i)

Mazda benchmark problem[2]**:**

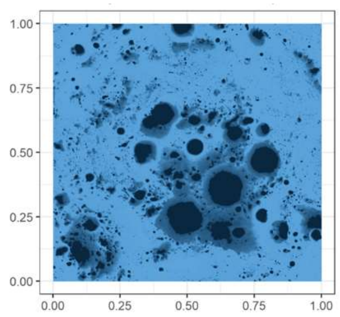
- Based on a real-world car structure design
- 222 design variables
- 2 objectives:
  - Minimization of the total weight of various cars
  - Maximization of the number of common gauge parts among various cars
- 54 constraints

[2] http://ladse.eng.isas.jaxa.jp/benchmark/

Real-World Test Problems Based on Simulations (ii)

Lunar lander landing site selection[3]**:**

- 2 design variables: coordinates x, y
- 3 objectives:
  - Total communication time
  - Continuous shade days
  - Landing point inclination angle
- 2 constraints
  - Max. continuous shade days
  - Max. landing point inclination angle



[3] http://www.jpnsec.org/files/competition2018/EC-Symposium-2018-Competition-English.html

Real-World Test Problems Based on Simulations (iii)

Wind turbine design problem[4]:

- Based on a real-world wind turbine design
- 32 design variables
- 5 objectives:
  - Annual power production
  - Average annual cost
  - Tower base load
  - Blade tip speed
  - Fatigue damage
- 22 constraints

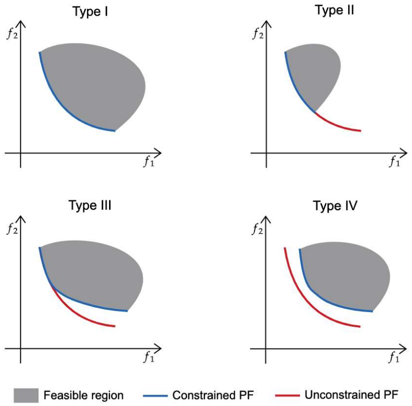
[4] http://www.jpnsec.org/files/competition2019/EC-Symposium-2019-Competition-English.html

Problem Characterization

Overview

- Type of CMOPs
- Pareto front shapes
- Problem landscapes

Type of CMOPs (Z. Ma et al. 2019)

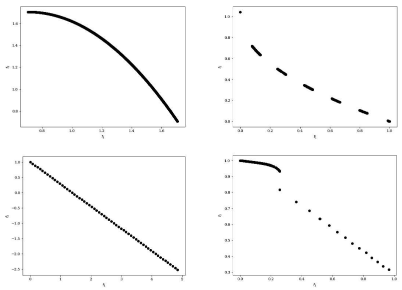


Test Suite Comparison: Type of CMOPs

Type	CTP	CF	C-DTLZ	NCTP	DC-DTLZ	DAS-CMOP	LIR-CMOP	MW
I	✓		✓	✓	✓	✓	✓	✓
II	✓	✓	✓		✓	✓	✓	✓
III		✓		✓		✓	✓	✓
IV	✓		✓	✓			✓	✓

Pareto Front Shapes

- Linear / Convex / Concave
- Connected / Disconnected / Discrete
- Mixed



Test Suite Comparison: Pareto Front Shapes

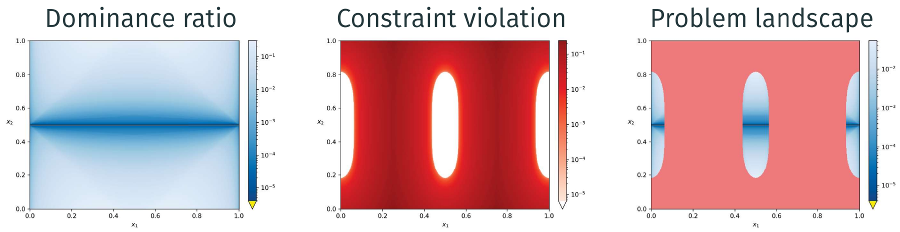
Type	CTP	CF	C-DTLZ	NCTP	DC-DTLZ	DAS-CMOP	LIR-CMOP	MW
Linear	✓	✓	✓	✓	✓	✓		✓
Convex	✓	✓	✓	✓		✓	✓	
Concave		✓	✓		✓	✓	✓	✓
Conn.	✓		✓	✓	✓	✓	✓	✓
Disconn.	✓	✓	✓		✓	✓	✓	✓
Discrete	✓	✓		✓		✓	✓	✓
Mixed	✓	✓		✓		✓	✓	✓

Problem Landscape

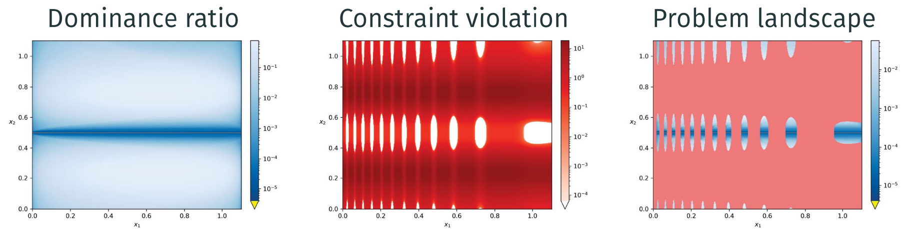
Constrained multiobjective problem landscape, $\mathcal{L}(S, f, v, d)$:

- $S \subseteq \mathbb{R}^n$ ... decision space
- $f : S \rightarrow \mathbb{R}^M$ ... objective vector function
- $v : S \rightarrow \mathbb{R}$ ... overall constraint violation function
- $d : S \times S \rightarrow \mathbb{R}$ ... distance metric

Example: C2-DTLZ2



Example: MW6



Exploratory Landscape Analysis: State of the Art

Multiobjective optimization:

- Limited studies in the multiobjective combinatorial context (Verel et al. 2013; Daolio et al. 2017; Liefooghe et al. 2020)
- Initial attempts to **visualize bi-objective continuous problems** (Fonseca 1995; Kerschke, Wang, et al. 2016; Kerschke and Grimme 2017; Schäpermeier et al. 2021)

Optimization in the presence of constraints:

- Preliminary study on the **characterization of constrained single-objective optimization problems** (Malan, Oberholzer, et al. 2015)
- Incorporation of these characteristics to **guide the constraint handling** (Malan 2018; Malan and Moser 2019)

Exploratory Landscape Analysis: Ongoing Research

Goals:

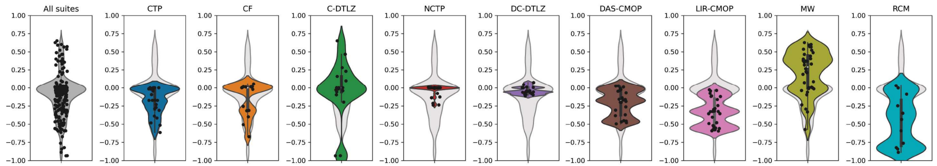
- Through assessment of the existing test suites of CMOPs
- Measuring **correlations between objectives and constraints**
- Identification of feasible **subregions and basins**

Methods:

- Space-filling design
- Random and adaptive walk
- Information content

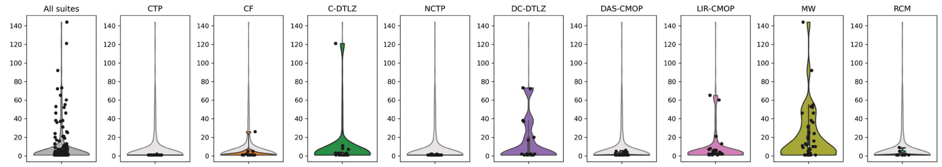
Test Suite Comparison: Correlations Between Objectives and Constraints

Correlations between objectives and constraints



Test Suite Comparison: Feasible Subregions

Number of feasible subregions



Note on Present Test Suites

- There are too **many Type I and Type II** CMOPs in the existing suites (Tanabe et al. 2017)
- Pareto front shapes of the artificial test problems are **unrealistically regular** (Ishibuchi et al. 2019)
- The existing artificial test problems have **many unrealistic characteristics**?

Performance Assessment

Performance Indicators

Any popular performance indicator for multiobjective optimization can be adapted for CMOPs by removing infeasible solutions

The most frequently used indicators in the literature are:

- Hypervolume (HV)
- Generational distance (GD and GD^+)
- Inverted generational distance (IGD and IGD^+)
- Epsilon indicator (EPS)

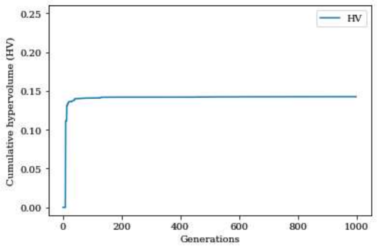
It is very important to use **cumulative indicators**: use all nondomnated feasible solutions found during the **entire run**

Constraint-Related Measures

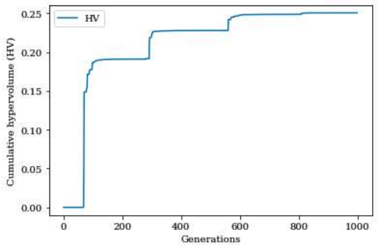
- Minimum of overall constraint violations
- Mean of overall constraint violations
- Feasibility ratio

Example: Hypervolume

MOEA/D-CDP

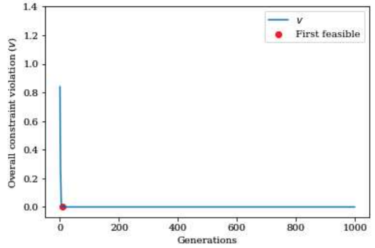


MOEA/D-IEpsilon

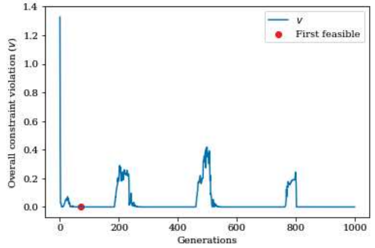


Example: Overall Constraint Violation

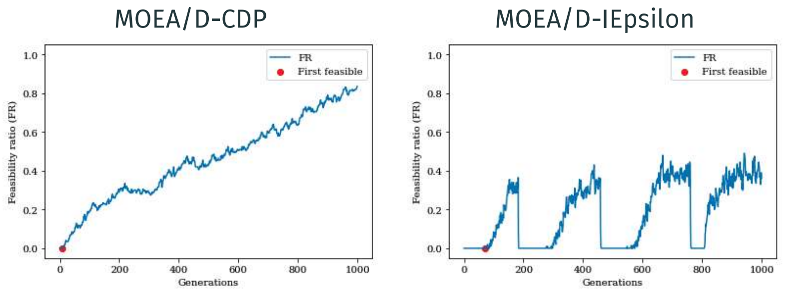
MOEA/D-CDP



MOEA/D-IEpsilon



Example: Feasibility Ratio



Software for Constrained Multiobjective Optimization

Overview

- Python
- R
- Matlab
- Java

Python (i)

pymoo: Multi-objective Optimization in Python[5]

- Algorithm implementations: NSGA-II (CDP), NSGA-III (CDP), C-TAEA
- CMOP implementations: TNK, OSY, BNH, CTP, DAS-CMOP, MW, 3 RCM problems
- Performance assessment: HV, GD, GD^+, IGD, IGD^+
- Additional: Solution repair when constraints are analytically expressed and visualization techniques

[5] https://pypi.org/project/pymoo/

Python (ii)

jMetalPy: Python Version of the JMetal Framework[6]

- Algorithm implementations: NSGA-II (CDP), NSGA-III (CDP), MOEAD-IEpsilon
- CMOP implementations: SRN, TNK, OSY, BNH, LIR-CMOP
- Performance assessment: HV, GD, IGD, EPS
- Additional: Statistical analysis and visualization techniques

[6] https://pypi.org/project/jmetalpy/

Python (iii)

deap: Distributed Evolutionary Algorithms in Python[7]

- Constraint handling by delta penalty approach, closest valid penalty approach, or island approach

pygmo: Parallel Optimization for Python[8]

- Advanced algorithms for hypervolume calculation

[7] https://pypi.org/project/deap/
[8] https://pypi.org/project/pygmo/

R

mco: Multiple Criteria Optimization Algorithms and Related Functions[9]

- Algorithm implementations: NSGA-II (CDP)
- CMOP implementations: BNH
- Performance assessment: HV, GD, EPS

MOEADr: Component-Wise MOEA/D Implementation[10]

- Constraint handling by penalty function approach, violation-based ranking
- Performance assessment: HV, IGD

[9] https://cran.r-project.org/web/packages/mco/index.html
[10] https://cran.r-project.org/web/packages/MOEADr/index.html

Matlab

PlatEMO: Evolutionary Multi-objective Optimization Platform[11]

- Algorithm implementations: NSGA-II (CDP), NSGA-III (CDP), ToP, PPS-MOEA/D, C-TAEA, MOEA/D-DAE, CCMO, CMOA-ES, MSCMO, POCEA
- CMOP implementations: CF, C-DTLZ, DC-DTLZ, LIR-CMOP, MW, DOC
- Performance assessment: HV, GD, IGD
- Additional: Statistical analysis, visualization techniques and GUI

[11] https://github.com/BIMK/PlatEMO

Java

MOEAFramework: A Free and Open Source Java Framework for Multiobjective Optimization[12]

- Algorithm implementations: NSGA-II (CDP)
- CMOP implementations: SRN, TNK, OSY, BNH, CF, C-DTLZ
- Performance assessment: HV, GD, IGD
- Additional: Statistical analysis and visualization techniques

jMetal: A Framework for Multi-objective Optimization with Metaheuristics[13]

- Same functionalities as jMetalPy

[12] https://github.com/MOEAFramework/MOEAFramework
[13] https://github.com/jMetal/jMetal

Conclusions

Summary

- Increasing interest in **constrained multiobjective optimization**
- Many new **techniques, test suites, and software** proposed in the last years
- **Problem characterization** is now gaining interest

Open Issues and Future Research Directions

- Advances in **exploratory landscape analysis** for CMOPs
- Artificial **test suites** reflecting real-world problem characteristics
- Comprehensive **algorithm performance assessment** in solving CMOPs
- **Assessment of the recently proposed CHTs** on real-world problems

Acknowledgment

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