SCALING AND VISUALIZING MULTIOBJECTIVE OPTIMIZATION TEST PROBLEMS WITH KNEES

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ABSTRACT

Knees in multiobjective optimization are regions on the Pareto optimal front where a small improvement in one objective leads to a large deterioration in at least one other objective. Without any additional knowledge on the preference of objectives the points on knees can be preferred to other points on the Pareto optimal front. Consequently, multiobjective optimization test problems with knees are very important as they can be used to test the algorithms' ability of finding solutions on the knees of a Pareto optimal front.

Two existing multiobjective optimization test problems with knees are DEB2DK and DEB3DK with two and three objectives, respectively. This paper introduces their scaled versions with four and five objectives. In addition, Pareto front approximations of these scaled problems are visualized using visualization with prosections. Besides properly visualizing the knees, prosections are able to maintain the dominance relations between most points in the approximation set.

1 INTRODUCTION

Solving a multiobjective optimization problem requires optimization of several conflicting criteria. As a consequence, the solution to such a problem is not a single point, but rather a set of points in the decision space, called *Pareto optimal set*, which corresponds to the *Pareto optimal front* in the objective space. In continuous optimization problems, Pareto optimal sets and fronts contain an infinite number of points.

Multiobjective optimization algorithms try to find a good finite approximation to the Pareto optimal front, called *approximation set*. Only mutually nondominated points are placed in the approximation set, where the dominance relation is defined as follows. Point *a dominates* point *b* if *a* is better than or equal to *b* in every objective and $a \neq b$. To aid the decision maker who selects the preferred solution among those from the approximation set, the points in the approximation sets should have the following properties:

– be located on the Pareto optimal front,
– have the maximum possible spread,
– be well distributed.

While the requirement of a good distribution of points in approximation sets most often translates into a uniform distribution, the desired distribution depends heavily on the shape of the Pareto optimal front. If, for example, the Pareto optimal front has *knees* (regions on the front where a small improvement in one objective leads to a large deterioration in at least one other objective) [2], it is more important that an approximation set contains points on these knees than some uniformly distributed points.

To test whether multiobjective algorithms are able to find points on the knees of Pareto optimal fronts, Branke et al. [2] presented a few benchmark optimization problems with knees (including DEB2DK and DEB3DK), which are based on the well known DTLZ test problem suite [4]. The problems were defined only for two and three objectives. This paper scales the definition of the DEB2DK and DEB3DK problems to four and five objectives and presents also the plots of their approximation sets using the recently proposed visualization technique called visualization with prosections [6, 7].

The rest of the paper is structured as follows. Section 2 presents the formal definition and some example plots of the DEB2DK and DEB3DK test problems, while their scaled versions are introduced in Section 3. Section 4 is dedicated to visualization in 4D and 5D, where prosections are used to visualize their approximation sets. The paper concludes with final remarks in Section 5.

2 DEB2DK AND DEB3DK

The DEB2DK and DEB3DK multiobjective optimization problems with knees are formally defined as:

DEB2DK

$$\min f_1(x) = g(x) r(x_1) \sin\left(\frac{\pi}{2} x_1\right)$$

$$\min f_2(x) = g(x) r(x_1) \cos\left(\frac{\pi}{2} x_1\right)$$

$$g(x) = 1 + \frac{9}{n-1} \sum_{i=2}^{n} x_i$$

$$r(x_1) = 5 + 10(x_1 - 0.5)^2 + \frac{\cos(2K\pi x_1)}{K}$$

$$0 \leq x_i \leq 1, i = 1, 2, \ldots, n$$

DEB3DK

$$\min f_1(x) = g(x)r(x)\sin\left(\frac{\pi}{2}x_1\right)\sin\left(\frac{\pi}{2}x_2\right)$$
$$\min f_2(x) = g(x)r(x)\sin\left(\frac{\pi}{2}x_1\right)\cos\left(\frac{\pi}{2}x_2\right)$$
$$\min f_3(x) = g(x)r(x)\cos\left(\frac{\pi}{2}x_1\right)$$
$$g(x) = 1 + \frac{9}{n-1}\sum_{i=2}^{n} x_i$$
$$r(x) = \frac{r_1(x_1) + r_2(x_2)}{2}$$
$$r_i(x_i) = 5 + 10(x_i - 0.5)^2 + \frac{2\cos(2K\pi x_i)}{K}$$
$$0 \le x_i \le 1, i = 1, 2, \ldots, n$$

Note that all objectives need to be minimized. In both problem definitions, n is the number of dimensions of the decision space and K is a parameter that together with the number of objectives m determines the number of knees in the Pareto front, K^{m-1}. Figure 1 shows the 2D problem with one (a) and three (b) knees, and the 3D problem with one (c) and four (d) knees[1].

In addition, we constructed approximation sets for the problems with one knee. Figure 2 (a) shows an approximation set consisting of 50 points from the known Pareto optimal front of the 2D problem, while Figure 2 (b) analogously presents an approximation set consisting of 500 points from the Pareto optimal front of the 3D problem. The single knee is clearly visible from both plots.

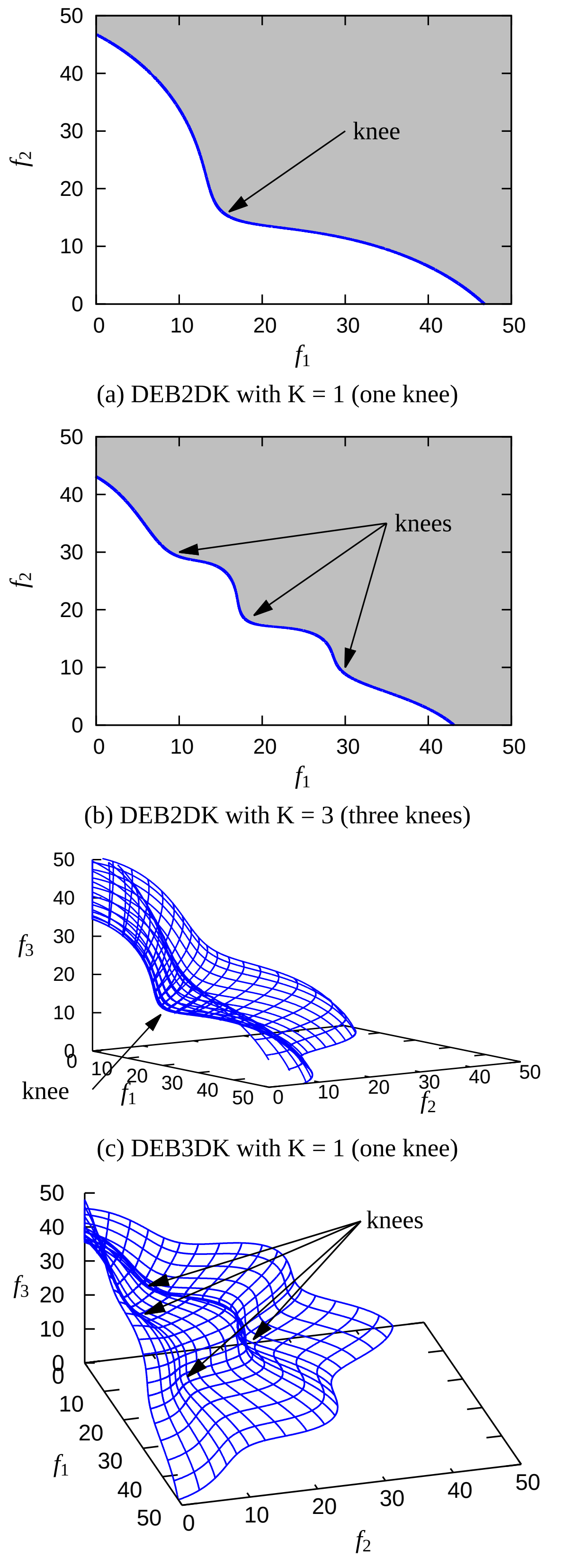
3 SCALING TO 4D AND 5D

As stated in [2], the DEB2DK and DEB3DK problems are based on the scalable DTLZ problems [4] and can thus be in turn scaled to any number of objectives. Here we present their scaled versions in four and five objectives, named DEB4DK and DEB5DK, respectively:

DEB4DK

$$\min f_1(x) = g(x)r(x)\sin\left(\frac{\pi}{2}x_1\right)\sin\left(\frac{\pi}{2}x_2\right)\sin\left(\frac{\pi}{2}x_3\right)$$
$$\min f_2(x) = g(x)r(x)\sin\left(\frac{\pi}{2}x_1\right)\sin\left(\frac{\pi}{2}x_2\right)\cos\left(\frac{\pi}{2}x_3\right)$$
$$\min f_3(x) = g(x)r(x)\sin\left(\frac{\pi}{2}x_1\right)\cos\left(\frac{\pi}{2}x_2\right)$$
$$\min f_4(x) = g(x)r(x)\cos\left(\frac{\pi}{2}x_1\right)$$
$$g(x) = 1 + \frac{9}{n-1}\sum_{i=2}^{n} x_i$$
$$r(x) = \frac{r_1(x_1) + r_2(x_2) + r_3(x_3)}{3}$$
$$r_i(x_i) = 5 + 10(x_i - 0.5)^2 + \frac{3\cos(2K\pi x_i)}{K}$$
$$0 \le x_i \le 1, i = 1, 2, \ldots, n$$

[1]The ranges of objectives of the DEB2DK and DEB3DK problems in this paper differ from the ones presented in [2]. This might be due to an unwanted integer division in the original implementation of these two problems (more specifically, in the calculation of the $g(x)$ function).



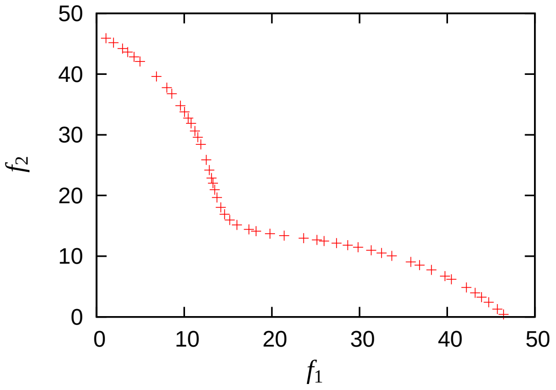
(a) DEB2DK with K = 1 (one knee)

(b) DEB2DK with K = 3 (three knees)

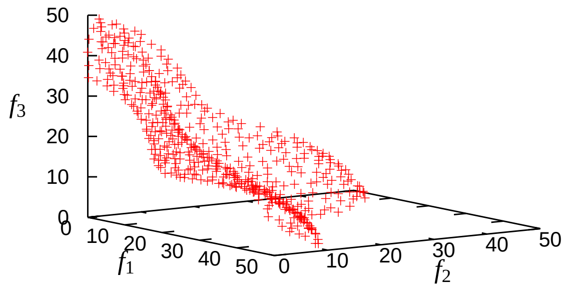
(c) DEB3DK with K = 1 (one knee)

(d) DEB3DK with K = 2 (four knees)

Figure 1: Pareto optimal fronts for different DEB2DK and DEB3DK problem instances.



(a) DEB2DK with K = 1



(b) DEB3DK with K = 1

Figure 2: Approximation sets of the DEB2DK and DEB3DK problems with one knee.

DEB5DK

$$\min f_1(x) = g(x)r(x)\sin\left(\frac{\pi}{2}x_1\right)\sin\left(\frac{\pi}{2}x_2\right)\sin\left(\frac{\pi}{2}x_3\right)\sin\left(\frac{\pi}{2}x_4\right)$$
$$\min f_2(x) = g(x)r(x)\sin\left(\frac{\pi}{2}x_1\right)\sin\left(\frac{\pi}{2}x_2\right)\sin\left(\frac{\pi}{2}x_3\right)\cos\left(\frac{\pi}{2}x_4\right)$$
$$\min f_3(x) = g(x)r(x)\sin\left(\frac{\pi}{2}x_1\right)\sin\left(\frac{\pi}{2}x_2\right)\cos\left(\frac{\pi}{2}x_3\right)$$
$$\min f_4(x) = g(x)r(x)\sin\left(\frac{\pi}{2}x_1\right)\cos\left(\frac{\pi}{2}x_2\right)$$
$$\min f_5(x) = g(x)r(x)\cos\left(\frac{\pi}{2}x_1\right)$$
$$g(x) = 1 + \frac{9}{n-1}\sum_{i=2}^{n} x_i$$
$$r(x) = \frac{r_1(x_1) + r_2(x_2) + r_3(x_3) + r_4(x_4)}{4}$$
$$r_i(x_i) = 5 + 10(x_i - 0.5)^2 + \frac{4\cos(2K\pi x_i)}{K}$$
$$0 \le x_i \le 1, i = 1, 2, \ldots, n$$

Again, n is the number of dimensions of the decision space and K controls the number of knees in the Pareto front. The scaling is performed by adding new objectives in spherical coordinates following the increment of dimension. Also, the $r(x)$ function is updated to include more individual $r_i(x_i)$ functions, while the constant in the $r_i(x_i)$ functions is modified to match the corresponding dimension. The only unaffected function is $g(x)$.

As in the DTLZ problems, the Pareto optimal fronts of all four problems are achieved when the solutions $x = (x_1, \ldots, x_{m-1}, x_m, \ldots, x_n)$ equal $(x_1, \ldots, x_{m-1}, 0.5, \ldots, 0.5)$.

4 VISUALIZING IN 4D AND 5D

There exist many methods for visualizing approximation sets in 4D and 5D (see [7] for a comprehensive review). However, only a few, for example, level diagrams [1], the hyper-radial visualization [3] and visualization with prosections [6, 7], are able to show knees of approximation sets. This paper presents the visualization of approximation sets of the scaled problems using prosections.

The term *prosection* means *pro*jection of a *section* [5] and the idea of visualization with prosections is to visualize only a section of the space at a time. All points that fall in this section are projected to one less dimension, while the others are discarded.

The section to be projected is defined using two parameters: angle φ and section width d (see Figure 3). If we wish to perform prosection on the plane $f_1 f_2$ with origin 0, the section is defined as $|f_1 \sin\varphi - f_2 \cos\varphi| \le d$. In the 4D case, all points in this section are mapped using the following function:

$$(f_1, f_2, f_3, f_4) \mapsto (f_1 \cos\varphi + f_2 \sin\varphi, f_3, f_4)$$

In the 5D case, prosection needs to be applied twice (the second time on the plane $f_3 f_4$). Using the angle ψ and section width e, the section now comprehends all points for which $|f_1 \sin\varphi - f_2 \cos\varphi| \le d$ and $|f_3 \sin\psi - f_4 \cos\psi| \le e$. These points are mapped using:

$$(f_1, f_2, f_3, f_4, f_5) \mapsto (f_1 \cos\varphi + f_2 \sin\varphi, f_3 \cos\psi + f_4 \sin\psi, f_5)$$

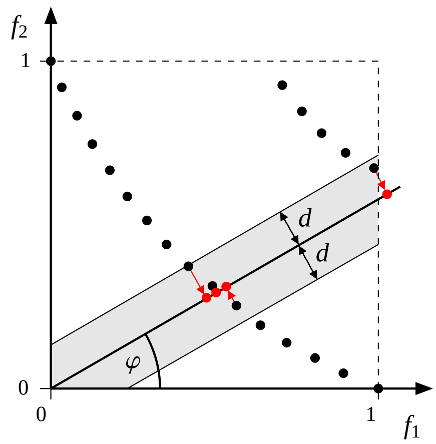
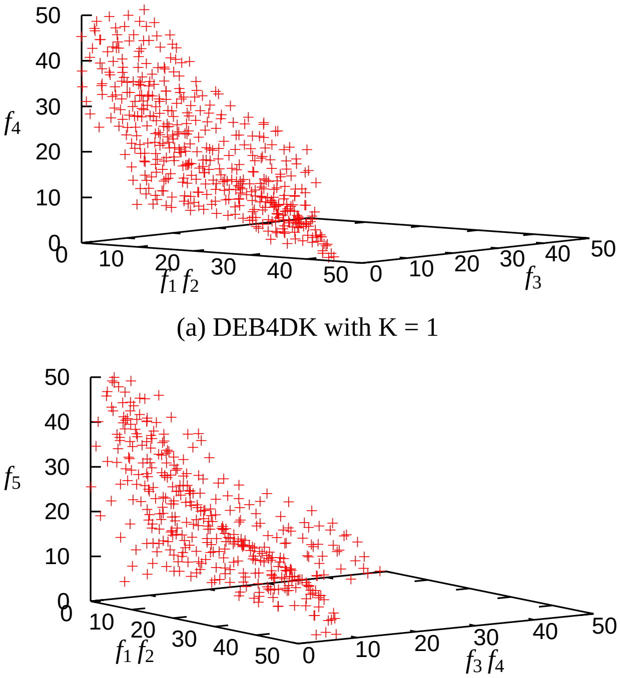


Figure 3: The section and projection used in the visualization with prosections.

The Pareto optimal fronts of the DEB4DK and DEB5DK test problems were sampled using 3,000 and 10,000 points, respectively. These approximation sets are visualized with prosections in Figure 4. In the 4D case, the angle $\varphi = 45°$ and section width $d = 2$ are used. In the 5D case, $\varphi = \psi = 45°$ and $d = e = 3$. As we can see, the knee is clearly visible in both plots, which resemble very much the DEB3DK plot from Figure 2 (b).



(a) DEB4DK with K = 1

(b) DEB5DK with K = 1

Figure 4: Prosections of approximation sets of the DEB4DK and DEB5DK test problems with one knee.

As formally proven in [7], besides maintaining well the shape of the visualized approximation set and the distribution of its points, prosections are able to preserve most of the dominance relations between vectors. This is especially important when comparing multiple approximation set—using prosections, several approximation sets can be visualized simultaneously.

The drawback of prosections is that only a portion of the space is visualized at a time. This requires to explore more than a single angle to gain a full understanding of an approximation set. See [7] for suggestions on how to tackle this issue.

5 CONCLUSION

In [2], the scalable DTLZ test problems were used to form two new multiobjective test problems with knees, called DEB2DK and DEB3DK. While declared scalable to multiple objectives, no formal definition of such scaled test problems was given. This paper presented the scaled problems in 4D and 5D, named DEB4DK and DEB5DK. As the Pareto optimal fronts of 4D and 5D problems cannot be simply plotted (in the way we did with DEB2DK and DEB3DK), we sampled the fronts to obtain two approximation sets. These were then visualized using prosections, which correctly showed the location of knees in the two approximation sets.

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